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Antoine Derighetti

Convolution Operators on Groups

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Preface

Roughly speaking a convolution operator T on a group G is a linear operator on complex functions $\varphi : G \to \mathbb{C}$ that commutes with left translations

$$_g(Tf) = T(_g f).$$

Typically convolution by fixed functions gives rise to convolution operators.

To be more precise, one has to specify G and the underlying function space for T. One may suppose that G is a locally compact group with Haar measure m, and choose T to be a continuous linear endomorphism of $L^p(G) = L^p(G; m)$, where $p > 1$ is some fixed real number. It's these convolution operators that will be the subject of this book, individual cases of them as well as, for given p and G, the space $CV_p(G)$ of all of them.

The set $CV_p(G)$ is a sub Banach algebra of the Banach algebra of all continuous linear endomorphisms of $L^p(G)$. If G is abelian, it is possible to define the Fourier transform of every T in $CV_2(G)$. The Fourier transform is a Banach algebra isometry of $CV_2(G)$ onto $L^\infty(\widehat{G})$. Here, $\widehat{G}$ denotes the Pontrjagin dual of G. Moreover, $CV_p(G) \subset CV_2(G)$, this permits to define the Fourier transform of every T in $CV_p(G)$.

The case of $G = \mathbb{R}^n$ involves results of classical Fourier analysis. For instance, the fact that the Heaviside function is the Fourier transform of some $T \in CV_p(\mathbb{R})$ implies Marcel Riesz's famous theorem on the convergence in L^p of Fourier series. This convergence still holds in two variables for square summation, but not for circular summation if $p \neq 2$. This reflects the fact that the indicator function of any square is the Fourier transform of some $T \in CV_p(\mathbb{R}^2)$ but not the indicator function of the disk except if $p = 2$.

In this book, we will be mainly concerned with the investigation of $CV_p(G)$ for noncommutative groups.

If $k \in L^p(G)$ and $l \in L^{p'}(G)$, then $\overline{k} * \check{l} \in C_0(G)$ with $\|\overline{k} * \check{l}\|_\infty \leq \|k\|_p \|l\|_{p'}$.

Forming series of such functions leads to the very important Figà-Talamanca space $A_p(G)$ contained in $C_0(G)$. $A_p(G)$ is an algebra for the pointwise product.

If it is given a norm based on $\|k\|_p \|l\|_{p'}$, it becomes a Banach algebra. There is a natural duality between $CV_p(G)$ and $A_p(G)$ for a large class of locally compact groups. This duality holds for all locally compact groups if $p = 2$. It is conjectured that it holds even for all p. If G is abelian, then $A_2(G)$ turns out to be the space of Fourier transforms of $L^1(\widehat{G})$. Here, again the Fourier transform is a Banach algebra isometry of $L^1(\widehat{G})$ onto $A_2(G)$.

To every integrable function on G, and more generally to every bounded measure on G, there corresponds by convolution an operator in $CV_p(G)$. For finite groups all of $CV_p(G)$ is obtained in this manner. It is not the case for infinite groups like $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{T}$ and probably for all infinite groups. Then we may ask whether every convolution operator may be approximated by operators associated to bounded measures, and in which topology. For $p = 2$ the answer is yes under the weak operator topology. This result was obtained by Murray and von Neumann for discrete groups, by Segal for unimodular groups and finally by Dixmier for general locally compact groups. The duality between $CV_p(G)$ and $A_p(G)$ permits to answer positively for $p \neq 2$ for all amenable groups.

Let I be an ideal of the algebra $A_p(G)$. The set of points of G where all functions in I vanish will be called the cospectrum of I. An elegant formulation of the celebrated tauberian theorem of Wiener is: if G is an abelian group every ideal of $A_2(G)$ with empty cospectrum is necessarily dense in $A_2(G)$. In this book, we will show that this statement holds for every group and also every $p > 1$. The fact that the theorem of Wiener is verified on arbitrary groups is highly surprising: there are papers suggesting the impossibility of such an extension for the group of two by two invertible matrices of complex numbers!

There is a hudge amount of literature concerning the non-commutative version of the Plancherel theorem and the inversion formula for C^∞ functions with compact support on Lie groups. Such questions are, for commutative groups, very simple. An achievement of this book is the extension to non-commutative groups of theorems which are deep and difficult even for $\mathbb{Z}$, $\mathbb{T}$ or $\mathbb{R}$.

An important part of this monograph deals with the relation between $CV_p(H)$ and $CV_p(G)$, where H is a closed subgroup of G. Let i be the inclusion map of H into G. Then i induces a canonical map, also denoted i, of $CV_p(H)$ into $CV_p(G)$. For $G = \mathbb{R}$ and $H = \mathbb{Z}$, this is a famous result due to Karel de Leeuw (1965), and to Saeki (1970) for G abelian and H arbitrary closed subgroup. It is also possible to characterise the image of i in $CV_p(G)$ and to obtain in this way non-commutative analogs of a result of Reiter (1963) concerning the relations between $L^\infty(\widehat{G})$ and $L^\infty(\widehat{H})$ and also to the fact that H is a set of synthesis in G (1956). The characterisation in $CV_p(G)$ of the image of $CV_p(H)$ under the map i, is a deep result due to Lohoué (1980). A large part of Chap. 7 is devoted to a detailed proof of Lohoué's result. As a consequence we obtain the extension of the Kaplansky–Helson theorem to non-abelian groups and to $p \neq 2$: for x in a arbitrary locally compact group G, every ideal of $A_p(G)$ having the cospectrum $\{x\}$ is dense in the set of all functions vanishing in x.

In the last chapter, we prove that for amenable groups $CV_p(G)$ is contained in $CV_2(G)$: this statement, compared to the commutative case, requires an entirely new approach.

The development of harmonic analysis on non-commutative groups is not just a straightforward generalization of the commutative case. It requires new ideas but it also gives rise to new problems which are far from being solved. For instance, the approximation theorem for non-amenable groups and for $p \neq 2$ is still out of reach. The investigation of the noncommutative case gives a better understanding of the commutative case! For example, instead of studying the relations between $L^\infty(\widehat{G})$ and $L^\infty(\widehat{H})$, it is more conceptual and more fruitfull to investigate the relations between the algebras $CV_2(G)$ and $CV_2(H)$.

A large part of the results presented appeared here for the first time in a book's form. The presentation is selfcontained and complete proofs are given. The prerequisities consists mostly with a familiarity with the books of Hewitt and Ross [66, 67]. (Chaps. 4, 6, 8 and 10), Reiter and Stegeman [105] and Rudin [107]. Notes at the end of the volume contain additional information about results of the text.

We wish to acknowledge our indebtedness to Professor Henri Joris, who read the proofs and helped to remove some errors and obscurities. His comments have stimulated us to improve the text in several places. Those errors which do appear in the text are, of course, my own responsibility. Thanks are also due to Professor Noël Lohoué and many colleagues for encouragement and help. We would like to thank especially Professor Gerhard Racher for improvements and suggestions in relation with chapter height.

Contents

List of Symbols

We list here the symbols which are systematically used throughout the book. The numbers in parentheses refer to the paragraphs where the symbols are defined.

$\mathcal{A}_p(G)$, $A_p(G)$ (3.1)

$[f]$, $\dot{f}$ (1.1.1)

$\check{\varphi}$, ${}_a\varphi$, φ_a, $\overline{\varphi}$, $\tilde{\varphi}$, φ^*, $\tau_p\varphi$ (1.1.2)

$[f]\check{}$, $\tau_p[f]$, ${}_a[f]$, $[f]_a$ (1.1.2)

$\check{\mu}$, $\overline{\mu}$, $\tilde{\mu}$ (1.1.2)

$\mathbb{F}_2$ (1.1.4)

α_p (1.5)

β (7.1)

$\varphi * \mu$ (1.1.3)

$CV_p(G)$ (1.2)

$\widehat{T}$ (1.6)

$C(X;Y)$, $C_{00}(X;Y)$, $C(X)$, $C_{00}(X)$, $C_0(X)$ (1.1.1)

$\mathcal{L}^p(X;\mu)$, $L^p(X;\mu)$, $\mathcal{L}^p(G)$, $L^p(G)$, $<>$ (1.1.1), (1.1.2)

$\mathcal{L}^p_V(X;\mu)$, $L^p_V(X;\mu)$ (3.3)

$\mathcal{L}(L^p(X;\mu))$ (1.1.1)

$\|T\|_p$ (1.1.1)

$\lambda^p_G(\mu)$, $\lambda^p_G(f)$, $\lambda^p_G([f])$ (1.2)

$\Lambda_{\hat{G}}$ (1.3)

$\mathcal{M}^\infty(X,\mu)$, $\mathcal{M}^\infty_{00}(X,\mu)$, $\mathcal{M}^\infty(G)$, $\mathcal{M}^\infty_{00}(G)$ (1.1.1), (1.1.2)

Chapter 1
Elementary Results

We give the basic properties of the Banach algebra $CV_p(G)$. For a locally compact abelian group G we show that $CV_2(G)$ is isomorphic to $L^\infty(\widehat{G})$ and define the Fourier transform of every element of $CV_p(G)$.

1.1 Basic Notations and Basic Definitions

1.1.1 Radon Measures and Integration Theory

Let X be a topological space and Y a topological vector space. We denote by $C(X;Y)$ the vector space of all continuous maps of X into Y and by $C_{00}(X;Y)$ the subspace of all maps having compact support. We put $C(X) = C(X;\mathbb{C})$ and $C_{00}(X) = C_{00}(X;\mathbb{C})$. We denote by $C_0(X)$ the subspace of all elements of $C(X)$ vanishing at infinity.

Suppose that X is a locally compact Hausdorff space and that μ is a complex Radon measure on X. For φ an arbitrary map of X into $[0,\infty]$

$$\int_X^* \varphi(x)d|\mu|(x)$$

denotes the upper integral in the sense of Bourbaki ([6], p. 112, Chap. IV, Sect. 4.1, no. 3, Définition 3.) We write $\mathcal{L}^1(X,\mu)$ for the $\mathbb{C}$-vector space of all $\varphi \in \mathbb{C}^X$ which are μ-integrable. For $\varphi \in \mathcal{L}^1(X,\mu)$ the integral of φ with respect to μ is denoted $\mu(\varphi)$ or

$$\int_X \varphi(x)d\mu(x).$$

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For $f \in \mathbb{C}^X$ or $[-\infty, \infty]^X$ locally μ-integrable we denote by $f\mu$ the Radon measure defined by

$$(f\mu)(\varphi) = \mu(f\varphi)$$

for every $\varphi \in C_{00}(X)$.

If $1 < p < \infty$ $\mathcal{L}^p(X, \mu)$ is the $\mathbb{C}$-vector space of all $\varphi \in \mathbb{C}^G$ such that φ is μ-measurable and $|\varphi|^p$ is μ-integrable. If $f \in \mathbb{C}^X$ we denote by $[f]$ the set of all $g \in \mathbb{C}^X$ with $g(x) = f(x)$ μ-almost everywhere and by $\dot{f}$ the set of all $g \in \mathbb{C}^X$ with $g(x) = f(x)$ locally μ-almost everywhere.

Suppose $1 \le p < \infty$. For $f \in \mathbb{C}^X$ or for an arbitrary map of X into $[-\infty, \infty]$ we put

$$N_p(f) = \left(\int_X^* |\varphi(x)|^p d|\mu|(x) \right)^{1/p} .$$

N_p is a semi-norm on $\mathcal{L}^p(X, \mu)$. With respect to this semi-norm $\mathcal{L}^p(X, \mu)$ is complete. For $f \in \mathcal{L}^p(X, \mu)$ we set

$$\|[f]\|_p = N_p(f) \quad \text{and} \quad L^p(X, \mu) = \left\{ [f] \,\middle|\, f \in \mathcal{L}^p(X, \mu) \right\}$$

which is a Banach space for the norm $\| \, \|_p$.

For an arbitrary map f of X into $[-\infty, \infty]$ we put

$$M_\infty(f) = \inf \left\{ \alpha \in [-\infty, \infty] \,\middle|\, f(x) \le \alpha \quad \text{locally } \mu\text{-almost everywhere} \right\}.$$

For $f \in \mathbb{C}^X$ we set $N_\infty(f) = M_\infty(|f|)$. For f a bounded complex function we set

$$\|f\|_u = \sup \left\{ |f(x)| \,\middle|\, x \in X \right\}.$$

Let $\mathcal{M}^\infty(X, \mu)$ be the $\mathbb{C}$-subspace of $\mathbb{C}^X$ of all functions which are μ-measurable and bounded and $\mathcal{L}^\infty(X, \mu)$ the $\mathbb{C}$-subspace of $\mathbb{C}^X$ of all functions which are locally μ-almost everywhere equal to a function of $\mathcal{M}^\infty(X, \mu)$. Then N_∞ is a semi-norm on $\mathcal{L}^\infty(X, \mu)$, with respect to this semi-norm $\mathcal{L}^\infty(X, \mu)$ is complete. By definition

$$L^\infty(X, \mu) = \left\{ \dot{f} \,\middle|\, f \in \mathcal{M}^\infty(X, \mu) \right\}.$$

With the norm

$$\|\dot{f}\|_\infty = N_\infty(f),$$

$L^\infty(X, \mu)$ is a Banach space. We denote by $\mathcal{M}^\infty_{00}(X, \mu)$ the subspace of all $f \in \mathcal{M}^\infty(X, \mu)$ with compact support.

Finally let $M^1(X)$ be the space of all complex bounded Radon measures on X. For $\mu \in M^1(X)$ we put

$$\|\mu\| = \sup\left\{|\mu(\varphi)| \,\middle|\, \varphi \in C_{00}(X), \|\varphi\|_u \leq 1\right\}.$$

Then $\| \, \|$ is a norm on $M^1(X)$.

Let $1 \leq p < \infty$ we put $p' = p/(p-1)$ if $p > 1$ and $p' = \infty$ if $p = 1$. For $f \in \mathcal{L}^p(X, \mu)$ and $g \in \mathcal{L}^{p'}(X, \mu)$ we set

$$\langle [f], [g] \rangle = \int_X f(x)\overline{g(x)} d\mu(x)$$

if $p > 1$, and if $p = 1$

$$\langle [f], \dot{g} \rangle = \int_X f(x)\overline{g(x)} d\mu(x).$$

The function $\langle , \rangle$ is a sesquilinear form on $L^p(X, \mu) \times L^{p'}(X, \mu)$.

Let $\mathcal{L}(L^p(X, \mu))$ be the linear space of all continuous endomorphisms of $L^p(X, \mu)$. For $T \in \mathcal{L}(L^p(X, \mu))$, $|||T|||_p$ is the bound of the operator T:

$$|||T|||_p = \sup\left\{\|Tf\|_p \,\middle|\, f \in L^p(X, \mu), \|f\|_p \leq 1\right\}.$$

For the composition of the operators, $\mathcal{L}(L^p(X, \mu))$ is a Banach algebra.

For V a topological vector space, V' denotes the dual of V. If $(V, \| \, \|_V)$ is a normed vector space, and if $F \in V'$ we put

$$\|F\|_{V'} = \sup\left\{|F(v)| \,\middle|\, v \in V, \|v\|_V \leq 1\right\}.$$

This norm makes V' into a Banach space.

1.1.2 Locally Compact Groups

Let G be a group. For a non-empty set Y, φ a map of G into Y, a and $x \in G$ we put

$$\check{\varphi}(x) = \varphi(x^{-1}), \ {}_a\varphi(x) = \varphi(ax) \quad \text{and} \quad \varphi_a(x) = \varphi(xa).$$

Let now be G a locally compact group. We always suppose that the topology of G is Hausdorff. We recall that there is a nonzero positive Radon measure m_G on G such that

$$m_G(\varphi) = m_G({}_a\varphi) = m_G(\varphi_a)\Delta_G(a) = m_G\big(\check{\varphi}\check{\Delta}_G\big)$$

for every $\varphi \in C_{00}(G)$ and every $a \in G$. Here Δ_G is a continuous homomorphism of G into the multiplicative group $(0, \infty)$. Up to a multiplicative real number, the measure m_G is unique. The measure m_G is called a left invariant Haar measure of G.

The function Δ_G does not depend of the choice of the measure m_G. This function is called the modular function of G. If G is compact we suppose that $m_G(1_G) = 1$.

Let us present some basic examples.

(a) If $G = \mathbb{T}$

$$m_{\mathbb{T}}(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta}) d\theta$$

for $\varphi \in C(\mathbb{T})$.

(b) For $G = \mathbb{R}$ we may choose

$$m_{\mathbb{R}}(\varphi) = \int_{-\infty}^{\infty} \varphi(x) dx$$

for every $\varphi \in C_{00}(\mathbb{R})$.

(c) Take now an arbitrary group G. Consider on G the discrete topology. This locally compact group is denoted G_d. Suppose at first that G is finite. Then $C_{00}(G_d) = \mathbb{C}^G$. We have

$$m_{G_d}(\varphi) = \frac{1}{|G|} \sum_{x \in G} \varphi(x)$$

for every $\varphi \in C(G_d)$. If G is infinite then $C_{00}(G_d)$ is the subspace of $\mathbb{C}^G$ of all functions having a finite support and we may choose

$$m_{G_d}(\varphi) = \sum_{x \in G} \varphi(x)$$

for every $\varphi \in C_{00}(G_d)$.

In all these examples $\Delta_G = 1$.

(d) Let G be the group of matrices

$$\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$$

where $x, y \in \mathbb{R}, x \neq 0$, with the topology induced by $\mathbb{R}^2$. Then we may choose

$$m_G(\varphi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi(x, y)}{x^2} dx dy.$$

One has

$$\Delta_G\left(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}\right) = \frac{1}{x^2}.$$

In the examples (a), (b) and (d), the integral on the right hand side is the Riemann integral.

Let again G be any locally compact group. For $\varphi \in \mathbb{C}^G$ we set

$$\overline{\varphi}(x) = \overline{\varphi(x)}, \quad \tilde{\varphi}(x) = \overline{\varphi(x^{-1})} \quad \text{and} \quad \varphi^*(x) = \overline{\varphi(x^{-1})}\Delta_G(x^{-1}).$$

Let be μ a complex Radon measure on G. Then we define the three Radon measures $\check{\mu}$, $\overline{\mu}$ and $\tilde{\mu}$ by

$$\check{\mu}(\varphi) = \mu(\check{\varphi}), \quad \overline{\mu}(\varphi) = \overline{\mu(\overline{\varphi})} \quad \text{and} \quad \tilde{\mu}(\varphi) = \overline{\mu(\tilde{\varphi})}$$

where $\varphi \in C_{00}(G)$. For $f \in \mathbb{C}^G$ and $1 \leq p < \infty$ we also put

$$\tau_p(f)(x) = f(x^{-1})\Delta_G(x^{-1})^{1/p}.$$

For f m_G-integrable we set

$$m_G(f) = m(f) = \int_G f(x)dx, \mathcal{L}^p(G) = \mathcal{L}^p(G, m_G), L^p(G) = L^p(G, m_G)$$
$$(1 \leq p \leq \infty)$$

and

$$\mathcal{M}_{00}^{\infty}(G) = \mathcal{M}_{00}^{\infty}(G, m_G).$$

For $f \in \mathbb{C}^G$ we put:

$$[f]^{\check{}} = [\check{f}] \quad \text{and for} \quad 1 \leq p < \infty \quad \tau_p[f] = [\tau_p f],$$

for $a \in G$ we also put

$${}_a[f] = [{}_a f] \quad \text{and} \quad [f]_a = [f_a].$$

Clearly τ_p is an isometric involution of the Banach space $L^p(G)$ for $1 \leq p < \infty$.

1.1.3 Convolution of Measures and Functions

Formally the convolution $\mu * \nu$ of the two Radon measures μ and ν on the locally compact group G is defined by

$$(\mu * \nu)(f) = \int_{G\times G} f(xy)d\mu(x)\,d\nu(y)$$

whenever the double integral converges absolutely for all $f \in C_{00}(G)$. This is the case for example if one of the two given measures has compact support, or if both of them are bounded. For $\mu = gm$ we have

$$(gm * \nu)(f) = \int_{G\times G} f(xy)g(x)dxd\nu(y) = \int_G \left(\int_G f(xy)g(x)dx \right) d\nu(y)$$
$$= \int_G \Delta_G(y^{-1}) \left(\int_G f(x)g(xy^{-1})dx \right) d\nu(y) = ((g * \nu)m)(f)$$

where we define

$$(g * \nu)(x) = \int_G g(xy^{-1})\Delta_G(y^{-1})d\nu(y).$$

Similarly we get $\mu * hm = (\mu * h)m$ if we define

$$(\mu * h)(y) = \int_G h(x^{-1}y)d\mu(x).$$

Putting here $\mu = gm$ we set $g * h = gm * h$ and thus

$$(g * h)(y) = \int_G g(yx)h(x^{-1})dx = \int_G g(yx^{-1})h(x)\Delta_G(x^{-1})dx.$$

We refer to [10] Chapter 8 for a detailed exposition of these questions.

1.1.4 Amenable Groups

A locally compact group G is said to be amenable if there is a linear functional $\mathcal{M}$ on the vector space $C^b(G)$ of all continuous bounded complex valued functions on G such that $\mathcal{M}(\varphi) \geqslant 0$ if $\varphi \geqslant 0$, $\mathcal{M}(1_G) = 1$ and $\mathcal{M}({}_a\varphi) = \mathcal{M}(\varphi)$ for every $a \in G$.

We only recall that compact, abelian or solvable groups are amenable. But $SL_2(\mathbb{R})$, the group of two by two real matrices with determinant one, and the free group $\mathbb{F}_2$ of two generators are not amenable. Every closed subgroup of an amenable group is amenable. If G is a locally compact group and H a closed normal subgroup, and if H and G/H are amenable, then so is G.

They are many properties equivalent to the amenability. We will use the following one: for every $\varepsilon > 0$ and for every compact subset K of G there is $s \in C_{00}(G)$ with $s \geqslant 0$, $N_1(s) = 1$ and $N_1({}_ks - s) < \varepsilon$ for every $k \in K$. We refer to Chap. 8 of [105] for detailed proofs of all these assertions.

1.2 Convolution Operators

Theorem 1. *Let G be a locally compact group,* $1 < p < \infty$, $\mu \in M^1(G)$ *and* $\varphi \in C_{00}(G)$. *For* $x \in G$ *we then have*

$$\left(\varphi * \Delta_G^{1/p'} \check{\mu}\right)(x) = \int_G \varphi(xy)\Delta_G(y)^{1/p}d\mu(y).$$

Moreover

$$\varphi * \Delta_G^{1/p'} \check{\mu} \in \mathcal{L}^p(G) \cap C(G) \quad \text{and} \quad N_p\left(\varphi * \Delta_G^{1/p'} \check{\mu}\right) \leq \|\mu\| N_p(\varphi).$$

Proof. We have

$$\begin{aligned}\left(\varphi * \Delta_G^{1/p'} \check{\mu}\right)(x) &= \int_G \varphi(xy^{-1})\Delta_G(y^{-1})d(\Delta_G^{1/p'} \check{\mu})(y) \\ &= \int_G \varphi(xy)\Delta_G(y)\Delta_G(y)^{-1/p'}d\mu(y) \\ &= \int_G \varphi(xy)\Delta_G(y)^{1/p}d\mu(y).\end{aligned}$$

Next we prove the continuity of $\varphi * \Delta_G^{1/p'} \check{\mu}$. Let $x_0 \in G$ and $\varepsilon > 0$. Choose U_0 a compact neighborhood of x_0 and $\eta > 0$ such that

$$\eta < \frac{\varepsilon}{\left(1 + \|\mu\|\right)\left(\sup_{y \in U_0^{-1} \operatorname{supp} \varphi} \Delta_G(y)^{1/p}\right)}.$$

There is an open neighborhood U_1 of e with

$$|\varphi(a) - \varphi(b)| < \eta$$

for every $a, b \in G$ with $ab^{-1} \in U_1$. Let U_2 be an open neighborhood of e with $U_2 \subset U_1$ and $U_2 x_0 \subset U_0$. Let $x \in U_2 x_0$ then for every $y \in G$ we have

$$|\varphi(xy) - \varphi(x_0 y)| \leq \eta \, 1_{U_0^{-1} \operatorname{supp} \varphi}(y).$$

Therefore

$$\left|(\varphi * \Delta_G^{1/p'} \check{\mu})(x) - (\varphi * \Delta_G^{1/p'} \check{\mu})(x_0)\right| \leq \int_G |\varphi(xy) - \varphi(x_0 y)|\Delta_G(y)^{1/p}d|\mu|(y) < \varepsilon.$$

To end the proof let ψ be any function in $C_{00}(G)$. We then have

$$\int_G \psi(x)(\varphi * \Delta_G^{1/p'} \check{\mu})(x)dx = \int_G \left(\int_G \varphi(xy)\Delta_G^{1/p}(y)\psi(x)dx\right) d\mu(y)$$

and therefore

$$\left|\int_G \psi(x)(\varphi * \Delta_G^{1/p'} \check{\mu})(x)dx\right| \leq \int_G N_{p'}\psi)N_p\big(\varphi_y \Delta_G(y)^{1/p}\big)d|\mu|(y)$$
$$= N_p(\varphi)N_{p'}(\psi)\|\mu\|.$$

Remark. The inequality

$$N_p(\varphi * \mu) \leq \|\mu\| N_p(\varphi)$$

is not verified in general. There is a missprint at (3.5.15) of page 108 of [105].

Corollary 2. *Let G and p as in Theorem 1. For every $\mu \in M^1(G)$ there is an unique $S \in \mathcal{L}(L^p(G))$ such that*

$$S[\varphi] = \left[\varphi * \left(\Delta_G^{1/p'} \check{\mu}\right)\right]$$

for every $\varphi \in C_{00}(G)$. For $f \in L^p(G)$ and $a \in G$ we have

$$_a(Sf) = S(_a f).$$

Moreover

$$|||S|||_p \leq \|\mu\|.$$

Proof. For a and $\varphi \in C_{00}(G)$ we have

$$_a\left(\varphi * \Delta_G^{1/p'} \check{\mu}\right) = \left(_a\varphi\right) * \Delta_G^{1/p'} \check{\mu}$$

and therefore

$$_a(S[\varphi]) = S(_a[\varphi]).$$

Definition 1. The operator S of Corollary 2 is denoted $\lambda_G^p(\mu)$.

Definition 2. For $f \in \mathcal{L}^1(G)$ we put $\lambda_G^p(f) = \lambda_G^p(fm)$ and $\lambda_G^p([f]) = \lambda_G^p(f)$.

In the following we will study a class of continuous operators of $L^p(G)$ which have the same essential property as the operator $\lambda_G^p(\mu)$.

Definition 3. Let G be a locally compact group and $1 < p < \infty$. An operator $T \in \mathcal{L}(L^p(G))$ is said to be a p-convolution operator of G if

$$T(_a\varphi) = {}_a(T(\varphi))$$

for every $a \in G$ and every $\varphi \in L^p(G)$. The set of all p-convolution operators of G is denoted $CV_p(G)$.

Proposition 3. *Let G be a locally compact group and $1 < p < \infty$. Then $CV_p(G)$ is a Banach subalgebra of $\mathcal{L}(L^p(G))$.*

Proposition 4. *Let G be a locally compact group and $1 < p < \infty$. Then:*

1. *λ_G^p is a linear injective contraction of the Banach space $M^1(G)$ into the Banach space $CV_p(G)$,*
2. *for every $a \in G$ and $\varphi \in L^p(G)$ we have*

$$\lambda_G^p(\delta_a)\varphi = \varphi_a \; \Delta_G(a)^{1/p}$$

and $\left\|\left|\lambda_G^p(\delta_a)\right|\right\|_p = 1$ where δ_a is the Dirac measure in a,

3. *$\lambda_G^p(\delta_{ab}) = \lambda_G^p(\delta_a)\lambda_G^p(\delta_b)$ for every $a, b \in G$,*
4. *for $f \in L^1(G)$ and $\varphi \in C_{00}(G)$ we have $\lambda_G^p(f)[\varphi] = [\varphi] * \tau_p f$.*

Remarks. 1. The map $x \mapsto \lambda_G^p(\delta_x)$ is an isometric representation of the locally compact group G into the Banach space $L^p(G)$. For $p = 2$ this map is called the right regular representation of G.
2. In *Sect. 4.1* we shall show that $\lambda_G^p(\alpha * \beta) = \lambda_G^p(\alpha)\lambda_G^p(\beta)$ for $\alpha, \beta \in M^1(G)$.

Theorem 5. *Let G be a locally compact group $1 < p < \infty$ and $T \in \mathcal{L}(L^p(G))$. Then $T \in CV_p(G)$ if and only if $T(f * \varphi) = f * T\varphi$ for every $f \in L^1(G)$ and every $\varphi \in L^p(G)$.*

Proof. We suppose that $T \in CV_p(G)$. Let $f \in C_{00}(G)$, $\varphi \in \mathcal{L}^p(G)$ and $\varphi_1 \in T[\varphi]$. We have $[f] * T[\varphi] = [f * \varphi_1]$ where for every $x \in G$

$$f * \varphi_1(x) = \int_G f(xy)\varphi_1(y^{-1})dy.$$

Let $\psi \in \mathcal{L}^{p'}(G)$. From

$$\int_G^* |f(y)| \left(\int_G |\varphi_1(y^{-1}x)||\psi(x)|dx \right) dy \leq N_p(\varphi_1)N_{p'}(\psi)N_1(f) < \infty,$$

we obtain

$$\langle [f] * T[\varphi], [\psi] \rangle = \int_G f(y) \left(\int_G \varphi_1(y^{-1}x)\overline{\psi(x)}dx \right) dy.$$

For every $y \in G$ we have

$$\int_G \varphi_1(y^{-1}x)\overline{\psi(x)}dx = \langle {}_{y^{-1}}T[\varphi], [\psi]\rangle = \langle T\ {}_{y^{-1}}[\varphi], [\psi]\rangle$$
$$= \langle {}_{y^{-1}}[\varphi], T^*[\psi]\rangle = \int_G \varphi(y^{-1}x)\,\overline{n(x)}dx$$

where T^* is the adjoint of T and where $n \in T^*[\psi]$. Therefore

$$\langle [f] * T[\varphi], [\psi]\rangle = \int_G f(y)\left(\int_G \varphi(y^{-1}x)\,\overline{n(x)}dx\right)dy.$$

We also have

$$\int_G^* |f(y)|\left(\int_G |\varphi(y^{-1}x)||n(x)|dx\right)dy \le N_p(\varphi)\,\|T^*[\psi]\|_{p'}N_1(f) < \infty,$$

and consequently

$$\int_G f(y)\left(\int_G \varphi(y^{-1}x)\,\overline{n(x)}dx\right)dy = \int_G\left(\int_G f(y)\varphi(y^{-1}x)dy\right)\overline{n(x)}dx.$$

This implies

$$\langle [f] * T[\varphi], [\psi]\rangle = \langle [f] * [\varphi], T^*[\psi]\rangle = \langle T([f] * [\varphi]), [\psi]\rangle$$

and therefore

$$T([f] * [\varphi]) = [f] * T[\varphi].$$

If $f \in \mathcal{L}^1(G)$, choose a sequence (f_n) of $C_{00}(G)$ with $N_1(f - f_n) \to 0$. By the inequality

$$\left\|T([f] * [\varphi]) - [f] * T[\varphi]\right\|_p \le |||T|||_p\, N_1(f - f_n)N_p(\varphi) + N_1(f_n - f)\left\|T[\varphi]\right\|_p$$

we obtain

$$T([f] * [\varphi]) = [f] * T[\varphi].$$

Now suppose that $T(f * \varphi) = f * T\varphi$ for every $f \in L^1(G)$ and every $\varphi \in L^p(G)$. Let $\varphi \in L^p(G)$, $a \in G$ and $\varepsilon > 0$. Choose $f \in L^1(G)$ such that

$$\|f * \varphi - \varphi\|_p < \frac{\varepsilon}{2(1 + |||T|||_p)}.$$

Then

$$\|T({}_a\varphi) - {}_a(T\varphi)\|_p \leqslant \|T({}_a\varphi) - T({}_a(f * \varphi))\|_p + \|T({}_a(f * \varphi)) - {}_a(T(f * \varphi))\|_p$$
$$+ \|{}_a(T(f * \varphi)) - {}_a(T\varphi)\|_p$$

But

$$T({}_a(f * \varphi)) = T(({}_a f) * \varphi)) = ({}_a f) * T\varphi = {}_a(f * T\varphi) = {}_a(T(f * \varphi))$$

and so

$$\|T({}_a\varphi) - {}_a(T\varphi)\|_p < \varepsilon.$$

Proposition 6. *Let G be a finite group. Then for $1 < p < \infty$ we have $CV_p(G) = \lambda_G^p(\mathbb{C}^G)$.*

Proof. Let $T \in CV_p(G)$. For $\varphi \in L^p(G)$ ($L^p(G) = \mathbb{C}^G$) we have $\varphi = \varphi * |G|1_{\{e\}}$ and $T\varphi = \varphi * f$ with $f = T(|G|1_{\{e\}})$. Then $T = \lambda_G^p(\check{f})$.

Remark. In general the calculation of $\left|\!\left|\!\left| \lambda_G^p(f) \right|\!\right|\!\right|_p$ is not easy.

We now present some examples of convolution operators on $G = \mathbb{Z}$ which are not of the type $\lambda_{\mathbb{Z}}^p(\mu)$ with $\mu \in M^1(\mathbb{Z})$.

Theorem 7. *Let f be the function on $\mathbb{Z}$ defined by*

$$f(n) = \frac{1}{n + \frac{1}{2}},\ n \in \mathbb{Z}.$$

For every $1 < p < \infty$ the map $\varphi \mapsto f * \varphi$ then belongs to $CV_p(Z)$.

Proof. See Titchmarsh [115] Theorem A.

Theorem 8. *Let f be the function on $\mathbb{Z}$ defined by*

$$f(n) = \frac{1}{n}, \quad n \in \mathbb{Z} \setminus \{0\}; f(0) = 0.$$

For $1 < p < \infty$ the map $\varphi \mapsto f * \varphi$ then belongs to $CV_p(Z)$.

Proof. See Riesz [106], Sect. 23 p. 241.

Remarks. 1. Clearly Theorems 7 and 8 are equivalent.
2. Let f be a non-negative function on $\mathbb{Z}$. If for some $1 < p < \infty$ the map $\varphi \mapsto f * \varphi$ belongs to $CV_p(\mathbb{Z})$, then $f \in l^1(\mathbb{Z})$ ([105], Theorem 8.3.10, p. 237). See the notes to Chap. 1.

3. For a certain class of locally compact abelian groups G, including $\mathbb{Z}$, $\mathbb{T}$ and $\mathbb{R}$, it is possible (and important) to improve the estimate

$$|||\lambda_G^p(f)|||_p \leq N_1(f)$$

for $f \in \mathcal{L}^1(G)$. See the book of Edwards and Gaudry [40] Chap. 2 Sects. 2.4, 2.4.4 Theorem p. 45.
4. For $G = SL_2(\mathbb{R})$ we have

$$|||\lambda_G^2(f)|||_2 \leq C_p N_p(f)$$

for every $f \in \mathcal{M}_{00}^{\infty}(G)$ and every $1 < p < 2$ (see Kunze and Stein [72] and the notes to Chap. 1 and to Chap. 3.)

1.3 For G Abelian $CV_2(G)$ is Isomorphic to $L^\infty(\widehat{G})$

Let $\widehat{G}$ be the dual group of a locally compact abelian group G. For $\mu \in M^1(G)$ the Fourier transform of μ is the function on $\widehat{G}$ defined by

$$\widehat{\mu}(\chi) = \int_G \overline{\chi(x)} d\mu(x).$$

The Fourier transform of $f \in \mathcal{L}^1(G)$ is the function $\widehat{f} = \widehat{fm_G}$:

$$\widehat{f}(\chi) = [\widehat{f}](\chi) = \int_G \overline{\chi(x)} f(x) dx.$$

We have $f \in C_0(\widehat{G})$. Let $m_{\hat{G}}$ be the unique Haar measure on $\widehat{G}$ such that for every $f \in \mathcal{L}^1(G) \cap C(G)$ with $\widehat{f} \in C_{00}(\widehat{G})$ we have

$$f(x) = \int_{\hat{G}} \widehat{f}(\chi)\chi(x) dm_{\hat{G}}(\chi)$$

for every $x \in G$. The measure $m_{\hat{G}}$ is said to be dual to the measure m_G. Let $\mathcal{F}$ be the unique continuous map of $L^2(G)$ into $L^2(\widehat{G})$ with $\mathcal{F}(f) = \widehat{f}$ for $f \in L^1(G) \cap L^2(G)$. We recall that $\mathcal{F}$ is an isometric isomorphism of the Banach space $L^2(G)$ onto $L^2(\widehat{G})$ and that for $f \in L^1(\widehat{G}) \cap L^2(\widehat{G})$ we have

$$\mathcal{F}^{-1}(f) = \left[\left(\widehat{f} \circ \varepsilon_G \right)^{\vee} \right]$$

where ε_G is the canonical map of G onto $\widehat{\widehat{G}}$.

Definition 1. Let G be a locally compact abelian group. For $f \in L^1(\widehat{G})$ we put

$$\Phi_{\hat{G}}(f) = (\widehat{f} \circ \varepsilon_G)\check{}.$$

Theorem 1. *Let G be a locally compact abelian group. Then the map $\Phi_{\hat{G}}$ is a contractive involutive monomorphism of the Banach algebra $L^1(\widehat{G})$ into $C_0(G)$.*

Definition 2. Let G be a locally compact abelian group. For $\varphi \in L^\infty(\widehat{G})$ we put

$$\Lambda_{\hat{G}}(\varphi)(f) = \mathcal{F}^{-1}\Big(\varphi\mathcal{F}(f)\Big)$$

where $f \in L^2(G)$.

Theorem 2. *Let G be a locally compact abelian group. Then $\Lambda_{\hat{G}}$ is an isometric involutive isomorphism of the Banach algebra $L^\infty(\widehat{G})$ onto the Banach algebra $CV_2(G)$.*

For $T \in CV_2(G)$ and $f \in L^2(\widehat{G})$ we have

$$\Lambda_{\hat{G}}^{-1}(T)f = \mathcal{F}(T(\mathcal{F}^{-1}(f))).$$

Proof. 1. $\Lambda_{\hat{G}}(\varphi) \in CV_2(G)$ and $\left|\!\left|\!\left|\Lambda_{\hat{G}}(\varphi)\right|\!\right|\!\right|_2 = \|\varphi\|_\infty$ for $\varphi \in L^\infty(\widehat{G})$.

Let $T = \Lambda_{\hat{G}}(\varphi)$ and $f \in L^2(G)$. We then have

$$\|T(f)\|_2 = \|\varphi\mathcal{F}(f)\|_2 \leq \|\varphi\|_\infty \, \|\mathcal{F}(f)\|_2 = \|\varphi\|_\infty \, \|f\|_2;$$

Therefore $T \in \mathcal{L}(L^2(G))$ and $|\!|\!|T|\!|\!|_2 \leq \|\varphi\|_\infty$.

For the proof of the reverse inequality, let $\eta > 0$ and $v \in \varphi$. There is $r \in C_{00}(\widehat{G})$ with $N_1(r) = 1$ and

$$\left|\int_{\hat{G}} r(\chi)v(\chi)d\chi\right| > \|\varphi\|_\infty - \eta.$$

But

$$\int_{\hat{G}} |r(\chi)||v(\chi)|d\chi \leq N_2(|r|^{1/2})N_2(v|r|^{1/2}) \leq |\!|\!|T|\!|\!|_2$$

and thus

$$\|\varphi\|_\infty < |\!|\!|T|\!|\!|_2 + \eta.$$

Next we verify that $T \in CV_2(G)$. For $a \in G$ and $f \in L^2(G)$ we have $\mathcal{F}({}_af) = \varepsilon_G(a)\mathcal{F}(f)$ and therefore

$$T({}_af) = \mathcal{F}^{-1}\Big(\varepsilon_G(a)\,(\varphi\mathcal{F}(f))\Big) = {}_a\Big(\mathcal{F}^{-1}(\varphi\mathcal{F}(f))\Big) = {}_aT(f).$$

2. $\Lambda_{\hat{G}}(\varphi\psi) = \Lambda_{\hat{G}}(\varphi)\Lambda_{\hat{G}}(\psi)$ for $\varphi, \psi \in L^\infty(\widehat{G})$.
   For $f \in L^2(G)$ we have

$$\Lambda_{\hat{G}}(\varphi\psi)(f) = \mathcal{F}^{-1}\big(\varphi(\psi\mathcal{F}(f))\big) = \mathcal{F}^{-1}\big(\varphi\,\mathcal{F}(\Lambda_{\hat{G}}(\psi)f)\big)$$
$$= \Lambda_{\hat{G}}(\varphi)(\Lambda_{\hat{G}}(\psi)(f)).$$

3. $\Lambda_{\hat{G}}(\overline{\varphi}) = \Lambda_{\hat{G}}(\varphi)^*$ for $\varphi \in L^\infty(\widehat{G})$.
   Let $f, g \in L^2(G)$. We have

$$\big\langle g, \Lambda_{\hat{G}}(\overline{\varphi})\big\rangle = \big\langle \mathcal{F}g, \overline{\varphi}\mathcal{F}(f)\big\rangle = \big\langle \varphi\mathcal{F}(g), \mathcal{F}(f)\big\rangle = \big\langle \Lambda_{\hat{G}}(\varphi)g, f\big\rangle$$
$$= \big\langle g, \Lambda_{\hat{G}}(\varphi)^* f\big\rangle.$$

4. $\Lambda_{\hat{G}}(L^\infty(\widehat{G})) = CV_2(G)$ and for $T \in CV_2(G)$ we have

$$\Lambda_{\hat{G}}^{-1}(T)f = \mathcal{F}\big(T(\mathcal{F}^{-1}(f))\big)$$

for every $f \in L^2(\widehat{G})$.
   Let $T \in CV_2(G)$. We put for $f \in L^2(\widehat{G})$

$$\Omega(f) = \mathcal{F}\big(T(\mathcal{F}^{-1}(f))\big),$$

Ω is a continuous linear operator of $L^2(\widehat{G})$. For $g \in L^1(G)$ we have

$$\mathcal{F}^{-1}([\hat{g}]f) = g * \mathcal{F}^{-1}(f)$$

(see [67] (31.27) Theorem p. 230) and therefore

$$\Omega([\hat{g}]f) = [\hat{g}]\Omega(f).$$

We now show that

$$\Omega([r]f) = [r]\Omega(f)$$

for every $r \in C_{00}(\widehat{G})$.
   Let $f_1, f_2 \in \mathcal{L}^2(\widehat{G})$ with $f_1 \in f$, $f_2 \in \Omega(f)$ and $\eta > 0$. There is $g \in L^1(G)$ (see [105], Chap. 5, Proposition 5.4.4, p. 161) with

$$\|\hat{g} - r\|_u < \frac{\eta}{2\big(1 + \|\!|\Omega|\!\|_2\big)\big(1 + N_2(f_1)\big)\big(1 + N_2(f_2)\big)}.$$

From

$$\|\Omega([r]f) - [r]\Omega(f)\|_2 \le \|\Omega([r]f) - \Omega([\hat{g}f_1])\|_2$$
$$+ \|\Omega([\hat{g}f_1]) - [\hat{g}]\Omega([f_1])\|_2 + \|[\hat{g}]\Omega([f_1]) - [r]\Omega(f)\|_2$$

we deduce

$$\|\Omega([r]f) - [r]\Omega(f)\|_2 \leq \|\|\Omega\|\|_2 \, N_2(rf_1 - \hat{g} f_1) + N_2(\hat{g} f_2 - r f_2)$$

and therefore

$$\|\Omega([r]f) - [r]\Omega(f)\|_2 < \eta.$$

According to [8] (Chap. II, Sect. 2.3, no. 3, Lemme 3, p. 145), there is $\varphi \in L^\infty(\widehat{G})$ with

$$\Omega(f) = \varphi f$$

for every $f \in L^2(\widehat{G})$. For $f \in L^2(G)$ we finally get

$$\Lambda_{\hat{G}}(\varphi) f = \mathcal{F}^{-1}\big(\Omega(\mathcal{F}(f))\big) = Tf.$$

Remarks. 1. For $T \in CV_2(G)$, φ and $\psi \in L^2(G)$ we have

$$\big\langle T\varphi, \psi \big\rangle = \Big\langle \Lambda_{\hat{G}}^{-1}(T)\mathcal{F}\varphi, \mathcal{F}\psi \Big\rangle.$$

2. Theorem 2 is already found in Larsen [73] (Chap. 4, p. 92, Theorem 4.1.1.).
3. For $\mu \in M^1(G)$ we have $\Lambda_{\hat{G}}^{-1}(\lambda_G^2(\mu)) = ((\widehat{\mu})\check{\ })^{\dot{}}$.

We also obtain an integral formula for the function Tf for many $T \in CV_2(G)$.

Corollary 3. *Let G be a locally compact abelian group and $T \in CV_2(G)$ with $\Lambda_{\hat{G}}^{-1}(T) \in L^1(\widehat{G})$. Then for $f \in L^2(G)$*

$$Tf = \Big[\Phi_{\hat{G}}\big(\Lambda_{\hat{G}}^{-1}(T)\big)\Big] * f.$$

Example. Theorem 2 implies that for $f \in \mathcal{L}^\infty(\mathbb{T})$, $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{C}$

$$\left| \sum_{j,k=1}^{n} x_j \overline{x_k} \widehat{f}(j-k) \right| \leq N_\infty(f) \sum_{j=1}^{n} |x_j|^2.$$

A special case of this result is due to Toeplitz ([116], p. 500, Satz 7). By Corollary 3 we also obtain for $T \in CV_2(\mathbb{Z})$ and $x \in l^2(\mathbb{Z})$

$$(Tx)(m) = \sum_{n=-\infty}^{\infty} \frac{x(n)}{2\pi} \int_0^{2\pi} \Lambda_{\hat{\mathbb{Z}}}^{-1}(T)(e^{i\theta}) e^{i(m-n)\theta} d\theta$$

([121], Zygmund, Vol. I, Chap. IV, Sect. 9, p. 168, (9.18) Theorem).

1.4 For G Abelian $CV_p(G)$ is Isomorphic to $CV_{p'}(G)$

Theorem 1. *Let G be a locally compact abelian group, $1 < p < \infty$ and $T \in CV_p(G)$. Then $T\varphi \in L^{p'}(G)$ for $\varphi \in L^p(G) \cap L^{p'}(G) \cap L^1(G)$ and $\|T\varphi\|_{p'} \leq |||T|||_p \|\varphi\|_{p'}$.*

Proof. Let $\varphi \in L^p(G) \cap L^{p'}(G) \cap L^1(G)$ and $\psi \in C_{00}(G)$. Let $r \in \varphi$ and $s \in T(\varphi)$. Then

$$\int_G s(x)\psi(x)dx = (s * \check{\psi})(e) = ((T\varphi) * [\psi]\check{})(e).$$

The group G being abelian, we have $(T\varphi) * [\psi]\check{} = [\psi]\check{} * (T\varphi)$. But according to Theorem 5 of Sect. 1.2,

$$[\psi]\check{} * (T\varphi) = T([\psi]\check{} * \varphi) = T(\varphi * [\psi]\check{}) = \varphi * T([\psi]\check{}),$$

and in particular $(T(\varphi) * [\psi]\check{})(e) = (\varphi * T([\psi]\check{}))(e)$ i.e.

$$\int_G s(x)\psi(x)dx = \int_G r(x)t(x^{-1})dx$$

where $t \in T([\psi]\check{})$. Therefore we get

$$\left| \int_G s(x)\psi(x)dx \right| \leq \|\varphi\|_{p'} \|[\psi]\|_p \, |||T|||_p$$

and consequently $T(\varphi) \in L^{p'}(G)$ with $\|T(\varphi)\|_{p'} \leq |||T|||_p \|\varphi\|_{p'}$.

Corollary 2. *Let G be a locally compact abelian group, $1 < p < \infty$ and $T \in CV_p(G)$. There is an unique continuous linear operator S of $L^{p'}(G)$ with $S\varphi = T\varphi$ for every $\varphi \in L^1(G) \cap L^p(G) \cap L^{p'}(G)$. We have $|||S|||_{p'} \leq |||T|||_p$.*

Definition 1. The unique continuous linear operator of the Banach space $L^{p'}(G)$ of Corollary 2 is denoted $j_p(T)$.

Proposition 3. *Let X be a locally compact Hausdorff space, μ a positive Radon measure on X, $(Y, \| \|)$ a normed space and f a μ-moderated and μ-measurable map of X into Y. Then there exists a sequence (g_n) of μ-measurable step functions with values in Y with following properties:*

1. $\|g_n(x)\| \leq \|f(x)\|$ *for every* $x \in X$ *and for every* $n \in \mathbb{N}$,
2. $\lim g_n(x) = f(x)$ *μ-almost everywhere,*
3. $\operatorname{supp} g_n$ *is compact.*

Proof. Suppose $f \neq 0$. There is $(K_n)_{n=1}^{\infty}$ a sequence of disjoint compact subsets of X and N μ-negligible with $\left\{x \middle| f(x) \neq 0\right\} = \left(\bigcup_{n=1}^{\infty} K_n\right) \cup N$. Moreover for every $n \in \mathbb{N}$ $K_n \neq \emptyset$ and $Res_{K_n} f \in C(K_n; Y)$. Let $i, n \in \mathbb{N}$ with $i \leqslant n$. For every $x \in K_i$, there is V, open neighborhood of x in the subspace K_i, such that

$$\|f(x) - f(x')\| < \frac{1}{2n}$$

for every $x' \in V$. There is therefore $k(i,n) \in \mathbb{N}$ and open subsets $V_1^{(i,n)}, \ldots, V_{k(i,n)}^{(i,n)}$ of K_i with $K_i = V_1^{(i,n)} \cup \cdots \cup V_{k(i,n)}^{(i,n)}$ and

$$\|f(x) - f(x')\| < \frac{1}{n}$$

for $x, x' \in V_j^{(i,n)}$ and $1 \leq j \leq k(i,n)$. Let $A_1^{(i,n)} = V_1^{(i,n)}$, for $2 \leq j \leq k(i,n)$ let $A_j^{(i,n)} = V_j^{(i,n)} \setminus \bigcup_{r=1}^{j-1} A_r^{(i,n)}$. The sets $A_1^{(i,n)}, \ldots, A_{k(i,n)}^{(i,n)}$ are disjoint Borel subsets of X with $K_i = \bigcup_{j=1}^{k(i,n)} A_r^{(i,n)}$. Let $I(i,n) = \left\{1 \leqslant j \leqslant k(i,n) \middle| A_j^{(i,n)} \neq \emptyset\right\}$. For every $j \in I(i,n)$ choose $x_{ij} \in A_j^{(i,n)}$.

If $x \in X \setminus \bigcup_{i=1}^{n} K_i$ we set $g_n(x) = 0$. If $x \in \bigcup_{i=1}^{n} K_i$ there is a unique $1 \leq i \leq n$ with $x \in K_i$ and a unique $j \in I(i,n)$ with $x \in A_j^{(i,n)}$. If $\|f(x_{ij})\| > \dfrac{1}{n}$ we set

$$g_n(x) = \left(1 - \frac{1}{n\|f(x_{ij})\|}\right) f(x_{ij}).$$

If $\|f(x_{ij})\| \leq \dfrac{1}{n}$ we put $g_n(x) = 0$. The sequence (g_n) has the required properties.

Remark. If $Y = \mathbb{C}$ then $g_n \in \mathcal{M}_{00}^{\infty}(X, \mu)$.

Lemma 4. *Let X be a locally compact Hausdorff space, μ a positive Radon measure on X, $r, s \in \mathbb{R}$, $a \in L^r(X,\mu)$ and $b \in L^s(X,\mu)$ with $1 \leq r < s$. Suppose the existence of a sequence $(f_n)_{n=1}^{\infty}$ of $L^r(X,\mu) \cap L^s(X,\mu)$ with*

$$\lim \|f_n - a\|_r = \lim \|f_n - b\|_s = 0.$$

Then $a = b$.

Theorem 5. *Let G be a locally compact abelian group and $1 < p < \infty$. Then j_p is an isometric isomorphism of the Banach algebra $CV_p(G)$ onto the Banach algebra $CV_{p'}(G)$.*

1. *For every $\mu \in M^1(G)$, we have $j_p(\lambda_G^p(\mu)) = \lambda_G^{p'}(\mu)$.*
2. *Let $T \in CV_p(G)$ and $\varphi \in L^p(G) \cap L^{p'}(G)$, then $j_p(T)\varphi = T\varphi$.*

Proof. Let $\varphi \in L^p(G) \cap L^{p'}(G)$. Consider $\varphi_1 \in \varphi$. Proposition 3 implies the existence of a sequence of m_G-measurable complex valued step functions (r_n) with $|r_n(x)| \le |\varphi_1(x)|$ for every $x \in X$ and $n \in \mathbb{N}$. We also have $\lim r_n(x) = \varphi_1(x)$ m_G-almost everywhere. But for every $n \in \mathbb{N}$ we have $r_n \in \mathcal{L}^p(G) \cap \mathcal{L}^{p'}(G) \cap \mathcal{L}^1(G)$. The Lebesgue 's theorem implies $\lim N_p(r_n - \varphi_1) = \lim N_{p'}(r_n - \varphi_1) = 0$. We get therefore

$$\lim \|T[r_n] - T\varphi\|_p = \lim \|j_p(T)[r_n] - j_p(T)\varphi\|_{p'} = 0.$$

Finally by Lemma 4 $T\varphi = j_p(T)\varphi$.

The following corollary improves Theorem 1.

Corollary 6. *Let G be a locally compact abelian group, $1 < p < \infty$ and $T \in CV_p(G)$. Then $T\varphi \in L^{p'}(G)$ for $\varphi \in L^p(G) \cap L^{p'}(G)$ and $\|T\varphi\|_{p'} \le \|\!|T|\!\|_p \|\varphi\|_{p'}$.*

Corollary 7. *Let G be a locally compact abelian group, $1 < p < \infty$ and $\mu \in M^1(G)$. Then*

$$\sup\left\{\left(\int_G \left|\int_G \varphi(xy)d\mu(y)\right|^p dx\right)^{1/p} \middle| \varphi \in C_{00}(G), N_p(\varphi) \le 1\right\}$$

$$= \sup\left\{\left(\int_G \left|\int_G \varphi(xy)d\mu(y)\right|^{p'} dx\right)^{1/p'} \middle| \varphi \in C_{00}(G), N_{p'}(\varphi) \le 1\right\}.$$

Remark. In 1976, Herz proved that for every finite nonabelian group G and for every $p \neq 2$, there is $f \in \mathbb{C}^G$ with $\left\|\!\left|\lambda_G^p(f)\right|\!\right\|_p \neq \left\|\!\left|\lambda_G^{p'}(f)\right|\!\right\|_{p'}$ i.e.

$$\sup\left\{\|f * g\|_p \middle| \|g\|_p \le 1\right\} \neq \sup\left\{\|f * g\|_{p'} \middle| \|g\|_{p'} \le 1\right\}$$

([63], Corollary 1, p. 12). See also the notes to Chap. 1.

1.5 $CV_p(G)$ as a Subspace of $CV_2(G)$

Theorem 1 (Riesz–Thorin). *Let μ be a positive Radon measure on a locally compact Hausdorff space X, and E the space of step functions in $L^1(X;\mu)$. Let $0 < \alpha \le \gamma \le 1$ and let the map $T : E \to L^{1/\alpha}(X;\mu) \cap L^{1/\gamma}(X;\mu)$ be linear and satisfy for $\varphi \in E$ the inequalities*

$$\|T\varphi\|_{1/\alpha} \leq M_1\|\varphi\|_{1/\alpha} \text{ and } \|T\varphi\|_{1/\gamma} \leq M_2\|\varphi\|_{1/\gamma}.$$

Then $T : E \to L^{1/\beta}(X;\mu)$ *for all* $\beta \in [\alpha,\gamma]$, *and*

$$\|\|T\|\|_{1/\beta} \leq M_1^{1-t}M_2^t \text{ where } \beta = (1-t)\alpha + t\gamma, 0 \leq t \leq 1.$$

Proof. Cf [67], Appendix E (E.18), p. 722 and (E.16), p. 719, Example (a).

Theorem 2. *Let* G *be a locally compact abelian group,* $1 < p < \infty$ *and* $T \in CV_p(G)$. *Then we have* $T\varphi \in L^2(G)$ *and* $\|T\varphi\|_2 \leq \|\|T\|\|_p \|\varphi\|_2$ *for every step function in* $L^1(G)$.

Proof. Clearly $\varphi \in L^1(G) \cap L^p(G) \cap L^{p'}(G)$. By Theorem 1 of Sect. 1.4 this implies $T\varphi \in L^{p'}(G)$ and $\|T\varphi\|_{p'} \leq \|\|T\|\|_p \|\varphi\|_{p'}$. We may suppose that $p \leq 2 \leq p'$. By Theorem 1, with $M_1 = M_2 = \|\|T\|\|_p$, we obtain immediately $T\varphi \in L^2(G)$ and $\|T\varphi\|_2 \leq \|\|T\|\|_p \|\varphi\|_2$.

Corollary 3. *Let* G *be a locally compact abelian group,* $1 < p < \infty$ *and* $T \in CV_p(G)$. *There is an unique* $S \in \mathcal{L}(L^2(G))$ *with* $S[r] = T[r]$ *for every integrable step function* r. *Moreover* $\|\|S\|\|_2 \leq \|\|T\|\|_p$.

Proof. It follows immediately from Theorem 2 and from the fact that the integrable step functions are dense in $L^p(G)$.

Definition 1. The unique operator of Corollary 3 is denoted $\alpha_p(T)$.

Theorem 4. *Let* G *be a locally compact abelian group and* $1 < p < \infty$. *Then* α_p *is a contractive algebra monomorphism of the Banach algebra* $CV_p(G)$ *into the Banach algebra* $CV_2(G)$. *For* $T \in CV_p(G)$ *and for* $\varphi \in L^p(G) \cap L^2(G)$ *we have* $\alpha_p(T)\varphi = T\varphi$. *Moreover for* $\mu \in M^1(G)$ *we have* $\alpha_p(\lambda_G^p(\mu)) = \lambda_G^2(\mu)$.

Proof. Let $T \in CV_p(G)$ and $\varphi \in L^p(G) \cap L^2(G)$ we show that $\alpha_p(T)\varphi = T\varphi$. Let $\varphi_1 \in \varphi$, there is a sequence of m_G-measurable complex valued step functions (r_n) with $\lim r_n(x) = \varphi_1(x)$ m_G-almost everywhere and $|r_n(x)| \leq |\varphi_1(x)|$ for every $n \in \mathbb{N}$ and for every $x \in G$. Then

$$\lim \|\alpha_p(T)[r_n] - \alpha_p(T)\varphi\|_2 = \lim \|T[r_n] - T\varphi\|_p = 0,$$

consequently $\alpha_p(T)\varphi = T\varphi$.

We obtain the following improvement of Theorem 2.

Corollary 5. *Let* G *be a locally compact abelian group,* $1 < p < \infty$ *and* $T \in CV_p(G)$. *For every* $\varphi \in L^p(G) \cap L^2(G)$ *we have* $T\varphi \in L^2(G)$ *and* $\|T\varphi\|_2 \leq \|\|T\|\|_p \|\varphi\|_2$.

Corollary 6. *Let G be a locally compact abelian group, $1 < p < \infty$ and $\mu \in M^1(G)$. Then*

$$\sup\left\{\left(\int_G \left|\int_G \varphi(xy)d\mu(y)\right|^2 dx\right)^{1/2} \middle| \varphi \in C_{00}(G), N_2(\varphi) \leq 1\right\}$$

$$\leq \sup\left\{\left(\int_G \left|\int_G \varphi(xy)d\mu(y)\right|^p dx\right)^{1/p} \middle| \varphi \in C_{00}(G), N_p(\varphi) \leq 1\right\}.$$

Remark. The proof of Corollary 5 (and also of Corollary 6) uses in a very strong way the commutativity of G. One the main result of this book is that Corollary 5 (and also Corollary 6) extends to the class of amenable groups.

1.6 The Fourier Transform of a Convolution Operator

Using the results and notations of Sects. 1.3 and 1.5, we introduce the Fourier transform of a p-convolution operator for arbitrary $p > 1$.

Definition 1. Let G be a locally compact abelian group and $1 < p < \infty$. The Fourier transform $\widehat{T} \in L^\infty(\widehat{G})$ of $T \in CV_p(G)$is defined by $\widehat{T} = \Lambda_{\hat{G}}^{-1}\big(\alpha_p(T)\big)$.

Theorem 1. *Let G be a locally compact abelian group and $1 < p < \infty$.*

1. $(S+T)\widehat{\ } = \widehat{S} + \widehat{T}$ *for* $S, T \in CV_p(G)$.
2. $(\alpha S)\widehat{\ } = \alpha\widehat{S}$ *for* $\alpha \in \mathbb{C}$ *and* $T \in CV_p(G)$.
3. $(ST)\widehat{\ } = \widehat{S}\widehat{T}$ *for* $S, T \in CV_p(G)$.
4. $\|\widehat{S}\|_\infty \leq |||S|||_p$ *for* $S \in CV_p(G)$.
5. *For* $S \in CV_p(G)$ *we have*

$$|||S|||_p = \sup\left\{\left|\left\langle \widehat{T}\mathcal{F}(\varphi), \mathcal{F}(\psi)\right\rangle\right| \middle| \varphi \in L^2(G) \cap L^p(G), \psi \in L^2(G) \cap L^{p'}(G),\right.$$
$$\left. \|\varphi\|_p \leq 1,\ \|\psi\|_{p'} \leq 1\right\}.$$

6. *For* $S \in CV_p(G)$ $\widehat{S} = 0$ *if and only if* $S = 0$.
7. $(\lambda_G^p(\mu))\widehat{\ } = ((\widehat{\mu}))\check{\ }$ *for every* $\mu \in M^1(G)$.
8. $(\lambda_G^p(\delta_a))\widehat{\ } = \varepsilon_G(a)\check{\ }$ *for every* $a \in G$.
9. *Let* $S \in CV_p(G)$ *such that* $\widehat{S} \in L^1(\widehat{G})$. *Then for* $\varphi \in L^2(G) \cap L^p(G)$ *we have for* $S\varphi$ *the integral formula* $S\varphi = [\Phi_{\hat{G}}(\widehat{S})] * \varphi$.

Proof. This Theorem is a consequence of Theorem 2 and Corollary 3 of Sect. 1.3 and Theorem 4 of Sect. 1.5.

Remark. For $T \in CV_p(\mathbb{R}^n)$ $\widehat{T}$ may be considered as a tempered distribution. For $\varphi \in C^\infty(\mathbb{R}^n)$ rapidly decreasing we then have $T\varphi = \mathcal{F}^{-1}(\widehat{T}) * \varphi$ $\mathcal{F}^{-1}(\widehat{T})$ being interpreted as the inverse Fourier transform of the distribution $\widehat{T}$.

Corollary 2. *Let $1 < p < \infty$ and $S \in CV_p(\mathbb{Z})$. If $p < 2$ suppose that $x \in l_p(\mathbb{Z})$, and $x \in l_2(\mathbb{Z})$ if $p \geq 2$. We then have*

$$(Sx)(m) = \sum_{n=-\infty}^{\infty} \frac{x(n)}{2\pi} \int_0^{2\pi} \widehat{S}(e^{i\theta}) e^{i(m-n)\theta} d\theta, \ m \in \mathbb{Z}.$$

The following proposition gives a characterization of the Fourier transform of a p-convolution operator.

Proposition 3. *Let G be a locally compact abelian group, $1 < p < \infty$, $\mathcal{E}$ a subspace of $L^2(G) \cap L^p(G)$ and $\mathcal{G}$ a subspace in $L^2(G) \cap L^{p'}(G)$. We suppose both spaces $\mathcal{E}$ and $\mathcal{G}$ invariant by translations ($f_a \in \mathcal{E}$ and $g_a \in \mathcal{G}$ for $f \in \mathcal{E}$ and $g \in \mathcal{G}$ and $a \in G$). We also suppose $\mathcal{E}$ dense in $L^2(G)$ and in $L^p(G)$ and similarly $\mathcal{F}$ dense in $L^2(G)$ and in $L^{p'}(G)$. Let also u be an element of $L^\infty(\widehat{G})$. The following statements are equivalent:*

1. *There is $S \in CV_p(G)$ such that $\widehat{S} = u$.*
2. *There is $K \in \mathbb{R}$ with $K > 0$ and*

$$\left|\left\langle u\mathcal{F}\varphi, \mathcal{F}\psi\right\rangle\right| \leq K\|\varphi\|_p\|\psi\|_{p'}$$

for every $\varphi \in \mathcal{E}$ and every $\psi \in \mathcal{G}$.

Moreover for $T \in CV_p(G)$ we have

$$|||T|||_p = \sup\left\{\left|\left\langle \widehat{T}\mathcal{F}\varphi, \mathcal{F}\psi\right\rangle\right| \,\middle|\, \varphi \in \mathcal{E}, \psi \in \mathcal{G}, \|\varphi\|_p \leqslant 1, \|\psi\|_{p'} \leqslant 1\right\}.$$

Proof. We show at first that (1) implies (2). Let $\varphi \in L^2(G) \cap L^p(G)$ and $\psi \in L^2(G) \cap L^{p'}(G)$. We have

$$\left|\left\langle u\mathcal{F}\varphi, \mathcal{F}\psi\right\rangle\right| = \left|\left\langle T\varphi, \psi\right\rangle\right| \leq |||T|||_p \|\varphi\|_p\|\psi\|_{p'}.$$

It remains to verify that (2) implies (1). There is a unique continuous sesquilinear map L of $L^p(G) \times L^{p'}(G)$ into $\mathbb{C}$ with

$$L(\varphi, \psi) = \left\langle u\mathcal{F}(\varphi), \mathcal{F}(\psi)\right\rangle$$

for every $\varphi \in \mathcal{E}$ and every $\psi \in \mathcal{G}$. There is therefore a unique $S \in \mathcal{L}(L^p(G))$ with

$$L(\varphi, \psi) = \langle S\varphi, \psi \rangle$$

for every $\varphi \in L^p(G)$ and every $\psi \in L^{p'}(G)$. It remains to verify that $S \in CV_p(G)$. Let $\varphi \in \mathcal{E}$, $\psi \in \mathcal{G}$ and $a \in G$. We have

$$\langle S({}_a\varphi), \psi \rangle = \langle u\,\mathcal{F}({}_a\varphi), \mathcal{F}(\psi) \rangle = \left\langle u\,(\varepsilon_G(a))^{\cdot}\mathcal{F}(\varphi), \mathcal{F}(\psi) \right\rangle = \left\langle u\,\mathcal{F}(\varphi), \mathcal{F}({}_{a^{-1}}\psi) \right\rangle$$

$$= \langle S\varphi, {}_{a^{-1}}\psi \rangle = \langle {}_a(S\varphi), \psi \rangle.$$

Corollary 4. *Let G be a locally compact abelian group, $1 < p < \infty$, a net (T_α) of $CV_p(G)$, $K \in (0, \infty)$ and $u \in L^\infty(\widehat{G})$. Suppose that:*

1. $|||T_\alpha|||_p \leq K$ *for every α,*
2. $\lim \widehat{T}_\alpha = u$ *for the weak topology $\sigma(L^\infty(\widehat{G}), L^1(\widehat{G}))$.*
   Then there is a $T \in CV_p(G)$ such that $\widehat{T} = u$. We have $|||T|||_p \leq K$.

Proof. Let $\varphi \in L^2(G) \cap L^p(G)$ and $\psi \in L^2(G) \cap L^{p'}(G)$. For every α we have

$$\langle T_\alpha\varphi, \psi \rangle = \left\langle \mathcal{F}(\varphi)\mathcal{F}(\tilde{\psi}), \overline{\widehat{T_\alpha}} \right\rangle.$$

But

$$\lim \left\langle \mathcal{F}(\varphi)\mathcal{F}(\tilde{\psi}), \overline{\widehat{T_\alpha}} \right\rangle = \left\langle \mathcal{F}(\varphi)\mathcal{F}(\tilde{\psi}), \overline{u} \right\rangle.$$

Consequently

$$\left|\langle u\,\mathcal{F}\varphi, \mathcal{F}\psi \rangle\right| \leq K\|\varphi\|_p\|\psi\|_{p'}.$$

Proposition 3 permits to finish the proof.

For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we put $\chi_m(e^{i\theta_1}, \ldots, e^{i\theta_n}) = e^{im_1\theta_1 + \cdots + im_n\theta_n}$. Then $\chi_m \in \widehat{\mathbb{T}^n}$. Let K be a compact neighborhood of 0 in $\mathbb{R}^n$. For $f \in L^1(\mathbb{T}^n)$ and $\lambda > 0$ we set

$$s_\lambda^{(K)} f = \sum_{m \in \mathbb{Z}^n \cap \lambda K} \widehat{f}(m)\chi_m.$$

Example. For $f \in L^1(\mathbb{T})$, $s_N^{([-1,1])} f$ is the Fourier sum of f.

The following theorem (see [111], p. 74, Teorema 4.1.) relates the L^p theory of Fourier series to $CV_p(\mathbb{R}^n)$.

Theorem 5. *Let K be a compact convex neighborhood of 0 in $\mathbb{R}^n$ and $1 < p < \infty$. The following statements are equivalent.*

1. $\lim_{\lambda\to\infty} \|f - s_\lambda^{(K)} f\|_p = 0$ *for every $f \in L^p(\mathbb{T}^n)$,*
2. *There is $T \in CV_p(\mathbb{R}^n)$ such that $\widehat{T} = [1_K]$.*

According to Marcel Riesz, $\lim_{\lambda\to\infty} \| f - s_\lambda^{([-1,1])} f \|_p = 0$ for every $f \in L^p(\mathbb{T})$ and for every $1 < p < \infty$. Consequently for every interval I of $\mathbb{R}$ and every $1 < p < \infty$ there is $T \in CV_p(\mathbb{R})$ with $\widehat{T} = [1_I]$. More generally for every $n > 1$, for every $p > 1$ and for every closed convex polyedral set C of $\mathbb{R}^n$ there is $T \in CV_p(\mathbb{R}^n)$ with $\widehat{T} = [1_C]$. But for D the unit ball in $\mathbb{R}^n$ ($n > 1$) and for $p \notin \left[2n/(n+1), 2n/(n-1)\right]$ there is no $T \in CV_p(\mathbb{R}^n)$ with $\widehat{T} = [1_D]$. Most of these results are due to Schwartz ([109], see also Herz [54]). Fefferman [43] proved that for every $p \neq 2$ and $n > 1$ there is no $T \in CV_p(\mathbb{R}^n)$ with $\widehat{T} = [1_D]$.

Chapter 2
The Commutation's Theorem

We show that for a locally compact unimodular group G, every $T \in CV_2(G)$ is the limit of convolution operators associated to bounded measures.

2.1 The Convolution Operator $T\lambda_G^p(f)$

Theorem 1. *Let G be a locally compact group, $1 < p < \infty$, $T \in CV_p(G)$, $f \in \mathcal{M}_{00}^{\infty}(G)$, $r \in T[f]$, $\varphi \in \mathcal{L}^p(G)$ and $\psi \in \mathcal{L}^{p'}(G)$. Then:*

1. $\psi * \tilde{r} \in \mathcal{L}^{p'}(G)$,
2. $N_{p'}(\psi * \tilde{r}) \leq |||T|||_p \, N_{p'}(\psi) \int_G |f(x)| \Delta_G(x)^{-\frac{1}{p'}} dx$,
3. $\left\langle T\lambda_G^p(\tau_p f)[\varphi], [\psi] \right\rangle = \int_G \varphi(x)\overline{(\psi * \tilde{r})(x)} dx.$

Proof. To begin with suppose $\varphi \in C_{00}(G)$. We have

$$\left\langle T\lambda_G^p(\tau_p f)[\varphi], [\psi] \right\rangle = \langle [\varphi * r], [\psi] \rangle.$$

From $(|\varphi| * |r|)|\psi| \in \mathcal{L}^1(G)$ we get

$$\int_G (\varphi * r)(x)\overline{\psi(x)} dx = \int_G \varphi(x)\overline{(\psi * \tilde{r})(x)} dx.$$

The inequalities

$$\left| \int_G \varphi(x)\overline{(\psi * \tilde{r})(x)} dx \right| \leq N_p(\varphi * r) N_{p'}(\psi) \leq |||T|||_p \, N_p(\varphi) N_{p'}(\psi) \int_G f(x) |\Delta_G(x)^{-\frac{1}{p'}} dx$$

prove (1) and (2).

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Suppose now that $\varphi \in \mathcal{L}^p(G)$.

There is a sequence (φ_n) of $C_{00}(G)$ with $N_p(\varphi_n - \varphi) \to 0$. We have

$$\lim \int_G \varphi_n(x)\overline{(\psi * \tilde{r})(x)}dx = \left\langle T\lambda_G^p(\tau_p f)[\varphi], [\psi] \right\rangle$$

and

$$\left| \int_G \varphi(x)\overline{(\psi * \tilde{r})(x)}dx - \int_G \varphi_n(x)\overline{(\psi * \tilde{r})(x)}dx \right| \le N_p(\varphi_n - \varphi)N_{p'}(\psi * \tilde{r}).$$

Consequently

$$\int_G \varphi(x)\overline{(\psi * \tilde{r})(x)}dx = \left\langle T\lambda_G^p(\tau_p f)[\varphi], [\psi] \right\rangle.$$

Remark. Even for $p = 2$, we are unable to decide whether $|\psi| * |\check{r}|$ is in $\mathcal{L}^{p'}(G)$.

We now show that every $T \in CV_p(G)$ can be approximated by $T\lambda_G^p(f)$.

Proposition 2. *Let G be a locally compact group, $1 < p < \infty$ and I the set of all $f \in C_{00}(G)$ with $f(x) \ge 0$ for every $x \in G$, $f(e) \ne 0$ and*

$$\int_G f(x)\Delta_G(x)^{-1/p'}dx = 1.$$

Then:

1. *on I the relation* supp $f' \subset$ supp f *is a filtering partial order,*
2. *for $f \in I$ we have $\left\|\left|\lambda_G^p(\tau_p f)\right|\right\|_p \le 1$,*
3. *for every $T \in CV_p(G)$ the net $\left(T\lambda_G^p(\tau_p f)\right)_{f \in I}$ converges strongly to T.*

Proof. Let $T \in CV_p(G)$, $\varphi \in \mathcal{L}^p(G)$ and $\varepsilon > 0$. Let U be a neighborhood of e in G such that for $y \in U$

$$N_p\left(\varphi - (\varphi)_{y^{-1}}\Delta_G(y^{-1})^{1/p}\right) < \frac{\varepsilon}{(1 + |||T|||_p)}.$$

Let also $f \in I$ with supp $f \subset U$. From

$$\|T[\varphi] - T\lambda_G^p(\tau_p f)[\varphi]\|_p \le |||T|||_p \int_G N_p\left(\varphi - (\varphi)_{y^{-1}}\Delta_G(y^{-1})^{1/p}\right) f(y)\Delta_G(y^{-1})^{1/p'}dy$$

we get $\|T[\varphi] - T\lambda_G^p(\tau_p f)[\varphi]\|_p < \varepsilon$.

The investigation of $CV_2(G)$ requires the study of those continuous operators S of $L^2(G)$ for which $S(\varphi_a) = (S\varphi)_a$.

In full analogy with Sect. 1.2 we have

$$(\mu * \varphi)(x) = \int_G \varphi(y^{-1}x)d\mu(y)$$

for $\mu \in M^1(G)$, $\varphi \in C_{00}(G)$ and $x \in G$. We also have $\mu * \varphi \in C(G) \cap \mathcal{L}^p(G)$ and

$$N_p(\mu * \varphi) \leq \|\mu\| N_p(\varphi)$$

for $1 < p < \infty$. There is a unique continuous operator S of $L^p(G)$ with $S[\varphi] = [\mu * \varphi]$ for $\varphi \in C_{00}(G)$. We have $S(f_a) = (Sf)_a$ for $f \in L^p(G)$ and $a \in G$. This operator S is denoted $\rho_G^p(\mu)$. For $f \in \mathcal{L}^1(G)$ we set $\rho_G^p(f) = \rho_G^p(f m_G)$ and $\rho_G^p([f]) = \rho_G^p(f)$.

Definition 1. Let G be a locally compact group, $1 < p < \infty$ and $S \in \mathcal{L}(L^p(G))$. We say that S belongs to the set $CV_p^d(G)$ if $S(\varphi_a) = (S\varphi)_a$ for every $a \in G$ and for every $\varphi \in L^p(G)$.

Proposition 3. *Let G be a locally compact group and $1 < p < \infty$. Then $CV_p^d(G)$ is a Banach subalgebra of $\mathcal{L}(L^p(G))$.*

Proposition 4. *Let G be a locally compact group and $1 < p < \infty$. Then:*

1. *ρ_G^p is a linear injective contraction of the Banach space $M^1(G)$ into the Banach space $CV_p^d(G)$,*
2. *for every $a \in G$ and every $\varphi \in L^p(G)$ we have*

$$\rho_G^p(\delta_a)\varphi = {}_{a^{-1}}\varphi$$

   and $\left\|\left\|\rho_G^p(\delta_a)\right\|\right\|_p = 1$,
3. *$\rho_G^p(\delta_{ab}) = \rho_G^p(\delta_a)\rho_G^p(\delta_b)$ for every $a, b \in G$,*
4. *for $f \in L^1(G)$ and $\varphi \in C_{00}(G)$ we have $\rho_G^p(f)[\varphi] = f * [\varphi]$.*

Theorem 5. *Let G be a locally compact group $1 < p < \infty$ and $S \in \mathcal{L}(L^p(G))$. Then $S \in CV_p^d(G)$ if and only if*

$$S\big(\varphi * \Delta_G^{1/p'} f\big) = (S\varphi) * \big(\Delta_G^{1/p'} f\big)$$

for every $f \in L^1(G)$ and every $\varphi \in L^p(G)$.

Remarks. 1. The map $x \mapsto \rho_G^2(\delta_x)$ is the left regular representation of G.

2. The proofs of Proposition 4 and Theorem 5 are entirely similar to those of the corresponding results concerning $CV_p(G)$ and λ_G^p (cf Sect. 1.2).

Similarly to Theorem 1 and Proposition 2 the following two results are verified.

Proposition 6. *Let G be a locally compact group, $1 < p < \infty$ and I the set of all $f \in C_{00}(G)$ with $f(x) \geq 0$ for every $x \in G$, $f(e) \neq 0$ and $\int_G f(x)dx = 1$. Then:*

1. *on I the relation $\operatorname{supp} f' \subset \operatorname{supp} f$ is a filtering partial order,*
2. *For $f \in I$ we have $\left\|\left|\rho_G^p(f)\right|\right\|_p \leq 1$,*
3. *For every $S \in CV_p^d(G)$ the net $\left(S\rho_G^p(f)\right)_{f\in I}$ converges strongly to S.*

Theorem 7. *Let G be a locally compact group, $1 < p < \infty$, $S \in CV_p^d(G)$, $g \in \mathcal{M}_{00}^{\infty}(G)$, $s \in S[g]$, $\varphi \in \mathcal{L}^p(G)$ and $\psi \in \mathcal{L}^{p'}(G)$. Then:*

1. *$s^* * \psi \in \mathcal{L}^{p'}(G)$,*
2. *$N_{p'}(s^* * \psi) \leq \|S\|_p N_{p'}(\psi)N_1(g)$,*
3. *$\left\langle S\rho_G^p(g)[\varphi], [\psi]\right\rangle = \int_G \varphi(x)\overline{(s^* * \psi)(x)}dx.$*

2.2 A Commutation Property of $CV_2(G)$

For $S \in CV_p^d(G)$ and $T \in CV_p(G)$, we first obtain integral formulas for $T\lambda_G^p(f)S\rho_G^p(g)$ and for $S\rho_G^p(g)T\lambda_G^p(f)$.

Proposition 1. *Let G be a locally compact group, $1 < p < \infty$, $S \in CV_p^d(G)$, $T \in CV_p(G)$, $f, g \in \mathcal{M}_{00}^{\infty}(G)$, $s \in S[g]$ and $r \in T[f]$. Then:*

$$\left\langle T\lambda_G^p(\tau_p f)S\rho_G^p(g)[\varphi], [\psi]\right\rangle = \int_G \varphi(x)\overline{(s^* * (\psi * \tilde{r}))(x)}dx$$

for $\varphi \in \mathcal{L}^p(G)$ and $\psi \in \mathcal{L}^{p'}(G)$.

Proof. Let $\varphi_1 \in S\rho_G^p(g)[\varphi]$ and

$$I = \left\langle T\lambda_G^p(\tau_p f)S\rho_G^p(g)[\varphi], [\psi]\right\rangle.$$

Then by Theorem 1 of Sect. 2.1 $\psi * \tilde{r} \in \mathcal{L}^{p'}(G)$ and $I = \int_G \varphi_1(x)\overline{(\psi * \tilde{r})(x)}dx.$

Consequently

$$I = \left\langle S\rho_G^p(g)[\varphi], [\psi * \tilde{r}]\right\rangle.$$

Then by Theorem 7 of Sect. 2.1 $s^* * (\psi * \tilde{r}) \in \mathcal{L}^{p'}(G)$ and

$$\left\langle S\rho_G^p(g)[\varphi], [\psi * \tilde{r}]\right\rangle = \int_G \varphi(x)\overline{(s^* * (\psi * \tilde{r}))(x)}dx.$$

Proposition 2. *Let G be a locally compact group, $1 < p < \infty$, $S \in CV_p^d(G)$, $T \in CV_p(G)$, $f, g \in \mathcal{M}_{00}^\infty(G)$, $s \in S[g]$ and $r \in T[f]$. Then:*

$$\left\langle S\rho_G^p(g)T\lambda_G^p(\tau_p f)[\varphi], [\psi]\right\rangle = \int_G \varphi(x)\overline{((s^* * \psi) * \tilde{r})(x)}dx$$

for every $\varphi \in \mathcal{L}^p(G)$ and every $\psi \in \mathcal{L}^{p'}(G)$.

Proof. Let $\varphi_1 \in T\lambda_G^p(\tau_p f)[\varphi]$ and

$$I = \left\langle S\rho_G^p(g)T\lambda_G^p(\tau_p f)[\varphi], [\psi]\right\rangle.$$

Then by Theorem 7 of Sect. 2.1 $s^* * \psi \in \mathcal{L}^{p'}(G)$ and

$$I = \int_G \varphi_1(x)\overline{(s^* * \psi)(x)}dx$$

and therefore

$$I = \left\langle T\lambda_G^p(\tau_p f)[\varphi], [s^* * \psi]\right\rangle.$$

We finally apply Theorem 1 of Sect. 2.1: we have $(s^* * \psi) * \tilde{r} \in \mathcal{L}^{p'}(G)$ and

$$I = \int_G \varphi(x)\overline{(s^* * (\psi * \tilde{r}))(x)}dx.$$

In the following it will be decisive to assume the unimodularity of the locally compact group G. With this assumption, we have $\tau_p f = \check{f}$.

Lemma 3. *Let G be a locally compact unimodular group, $S \in CV_2^d(G)$, $T \in CV_2(G)$ and $f, g \in \mathcal{M}_{00}^\infty(G)$. Then $T\lambda_G^2(f)S\rho_G^2(g) = S\rho_G^2(g)T\lambda_G^2(f)$.*

Proof. For $r \in T[f]$, $s \in S[g]$ and $\varphi, \psi \in \mathcal{M}_{00}^\infty(G)$ we have

$$\left\langle T\lambda_G^2(\check{f})S\rho_G^2(g)[\varphi], [\psi]\right\rangle = \int_G \varphi(x)\overline{(s^* * (\psi * \tilde{r}))(x)}dx$$

and

$$\left\langle S\rho_G^2(g)T\lambda_G^2(\check{f})[\varphi], [\psi]\right\rangle = \int_G \varphi(x)\overline{((s^* * \psi) * \tilde{r})(x)}dx.$$

By the unimodularity of G, for every $x \in G$ we have $(|\psi| * |\check{r}|)_x \in \mathcal{L}^2(G)$, and consequently

$$\int_G^* |s(y)| \left(\int_G |\psi(yxz)| r(z)| dz \right) dy < \infty.$$

This implies $s^* * (\psi * \tilde{r}) = (s^* * \psi) * \tilde{r}$.

Theorem 4. *Let G be a locally compact unimodular group. Then $ST = TS$ for $S \in CV_2^d(G)$ and $T \in CV_2(G)$.*

Proof. To begin with we prove that for $S \in CV_2^d(G)$, $T \in CV_2(G)$ and $f \in \mathcal{M}_{00}^{\infty}(G)$ we have $ST\lambda_G^2(f) = T\lambda_G^2(f)S$.

Let $\varphi \in L^2(G)$ and $\varepsilon > 0$. There is $g \in C_{00}(G)$ with:

$g(x) \geq 0$ for every $x \in G$, $\int_G g(x)dx = 1$,

$$\|S\rho_G^2(g)\varphi - S\varphi\|_2 < \frac{\varepsilon}{2(1 + \|T\lambda_G^2(f)\|_2)} \quad \text{and} \quad \|S\rho_G^2(g)T\lambda_G^2(f)\varphi - ST\lambda_G^2(f)\varphi\|_2 < \frac{\varepsilon}{2}.$$

Now from

$$\|ST\lambda_G^2(f)\varphi - T\lambda_G^2(f)S\varphi\|_2 \leq \|ST\lambda_G^2(f)\varphi - S\rho_G^2(g)T\lambda_G^2(f)\varphi\|_2$$

$$+\|S\rho_G^2(g)T\lambda_G^2(f)\varphi - T\lambda_G^2(f)S\rho_G^2(g)\varphi\|_2 + \|T\lambda_G^2(f)S\rho_G^2(g)\varphi - T\lambda_G^2(f)S\varphi\|_2,$$

Lemma 3 and

$$\|T\lambda_G^2(f)S\rho_G^2(g)\varphi - T\lambda_G^2(f)S\varphi\|_2 < \frac{\varepsilon}{2}$$

we get

$$\|ST\lambda_G^2(f)\varphi - T\lambda_G^2(f)S\varphi\|_2 < \varepsilon.$$

Next let $\varphi \in L^2(G)$ and $\varepsilon > 0$. According to Proposition 2 of Sect. 2.1 there is $f \in C_{00}(G)$ with $f(x) \geq 0$ for every $x \in G$, $\int_G f(x)dx = 1$,

$$\|TS\varphi - T\lambda_G^2(f)S\varphi\|_2 < \frac{\varepsilon}{2} \quad \text{and} \quad \|T\varphi - T\lambda_G^2(f)\varphi\|_2 < \frac{\varepsilon}{2(1 + \|S\|_2)}.$$

From

$$\|TS\varphi - ST\varphi\|_2$$

$$\leq \|TS\varphi - T\lambda_G^2(f)S\varphi\|_2 + \|T\lambda_G^2(f)S\varphi - ST\lambda_G^2(f)\varphi\|_2 + \|ST\lambda_G^2(f)\varphi - ST\varphi\|_2,$$

$$T\lambda_G^2(f)S\varphi = ST\lambda_G^2(f)\varphi \quad \text{and} \quad \|ST\lambda_G^2(f)\varphi - ST\varphi\|_2 < \frac{\varepsilon}{2}$$

we obtain

$$\|TS\varphi - ST\varphi\|_2 < \varepsilon.$$

2.3 An Approximation Theorem for $CV_2(G)$

Using the commutation theorem of Sect. 2.2 (Theorem 4) we show that every $T \in CV_2(G)$ is the limit of $\lambda_G^2(\mu)$ for G a locally compact unimodular group.

For a complex Hilbert space $\mathcal{H}$, we denote by $\mathcal{L}(\mathcal{H})$ the involutive Banach algebra of all continuous operators of $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, $\|T\|$ is the norm of the operator T. For $\mathcal{E}$ a subset of $\mathcal{L}(\mathcal{H})$ we denote by $\mathcal{E}'$ the set of all $T \in \mathcal{L}(\mathcal{H})$ with $ST = TS$ for every $S \in \mathcal{E}$, and we put $\mathcal{E}'' = (\mathcal{E}')'$.

Theorem 1. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}$ an involutive subalgebra of $\mathcal{L}(\mathcal{H})$ with $\left\{Tx \middle| x \in H, T \in \mathcal{B}\right\}$ dense in H. Then $\mathcal{B}''$ coincides with the closure of $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$ with respect to the strong operator topology.*

Proof. See [36], J. Dixmier, Chap. I, Sect. 3, no. 4, Corollaire 1, p. 42.

The next result is Kaplansky's density theorem.

Theorem 2. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}, \mathcal{C}$ two involutive subalgebras of $\mathcal{L}(\mathcal{H})$ with $\mathcal{B} \subset \mathcal{C}$. Suppose that $\mathcal{C}$ is dense in the strong closure of $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$. Then for every $T \in \mathcal{C}$ there is a net (S_α) of $\mathcal{B}$ such that:*

1. $\lim_\alpha S_\alpha = T$ *strongly,*
2. $\|S_\alpha\| \leq \|T\|$ *for every* α.

Proof. See Dixmier, [36], Chap. I, Sect. 3, no. 5, Théorème 3, p. 43–44.

Let G be a locally compact group. In this paragraph, we denote by $\mathcal{A}$ the set of all $\lambda_G^2(\mu)$, where μ is a complex measure with finite support. Clearly $\mathcal{A}$ is an involutive subalgebra $\mathcal{L}(L^2(G))$ with unit: $\lambda_G^2(\mu)^* = \lambda_G^2(\tilde{\mu})$ and $\lambda_G^2(\delta_e) = \mathrm{id}_{L^2_{\mathbb{C}}(G)}$. The following statement is straightforward.

Proposition 3. *Let G be a locally compact group. Then $CV_2^d(G) = \mathcal{A}'$.*

We obtain now the promised approximation theorem for $CV_2(G)$.

Theorem 4. *Let G be a locally compact unimodular group and $T \in CV_2(G)$. There is a net (μ_α) of complex measures with finite support such that:*

1. $\lim_\alpha \lambda_G^2(\mu_\alpha) = T$ *strongly,*
2. $\left|\!\left|\!\left|\lambda_G^2(\mu_\alpha)\right|\!\right|\!\right|_2 \leq |\!|\!|T|\!|\!|_2$ *for every* α.

Proof. By Theorem 4 of Sect. 2.2 we have $T \in \mathcal{A}''$. It suffices to apply Theorems 1 and 2 to finish the proof.

Remarks. 1. The fact that $\left\{\lambda_G^2(\delta_x) \middle| x \in G\right\}'' = CV_2(G)$, for G locally compact and unimodular, is due to Segal ([110], Theorem, p. 294). The case of G discrete, was obtained earlier by Murray and von Neumann ([96], Lemma 5.3.3, p. 789).

2. Using different methods, Dixmier obtained $\left\{\lambda_G^2(\delta_x) \middle| x \in G\right\}'' = CV_2(G)$, and consequently Theorem 4, for every locally compact group G ([35], Théorème 1,

p. 280, [36], Chap. I, Sect. 5, p. 71, Théorème 1 and Exercice 5 p. 80). See also Mackey ([90], p. 207, Lemma 3.3.)

Theorem 5. *Let G be a locally compact unimodular group and $T \in CV_2(G)$. There is a net (f_α) of $C_{00}(G)$ such that:*

1. $\lim_\alpha \lambda_G^2(f_\alpha) = T$ *strongly,*
2. $\left\| \lambda_G^2(f_\alpha) \right\|_2 \leq \| T \|_2$ *for every* α.

Proof. According to Theorem 1 $\left(\lambda_G^2(C_{00}(G))\right)''$ is the strong closure of $\lambda_G^2(C_{00}(G))$. But by Theorem 5 of Sect. 2.1 $\left(\lambda_G^2(C_{00}(G))\right)' = CV_2^d(G)$ and consequently

$$\left(\lambda_G^2(C_{00}(G))\right)'' = CV_2(G).$$

Remark. We will extend this result to $p \neq 2$ for certain classes of locally compact groups. We will also try to give more information on the approximating net (f_α).

Chapter 3
The Figa–Talamanca Herz Algebra

Let G be a locally compact group. The Banach space $A_p(G)$, generated by the coefficients of the regular representation in $L^{p'}(G)$, is a Banach algebra for the pointwise product on G. If G is abelian then $A_2(G)$ is isomorphic to $L^1(\widehat{G})$.

3.1 Definition of $A_p(G)$

Let G be a locally compact group and $1 < p < \infty$. For $k \in \mathcal{L}^p(G)$ and $l \in \mathcal{L}^{p'}(G)$ we have $\overline{k} * \check{l} \in C_0(G)$. The function $\overline{k} * \check{l}$ is a coefficient of the right regular representation in $L^{p'}(G)$:

$$\overline{k} * \check{l}(x) = \left\langle \lambda_G^{p'}(\delta_x)[\tau_{p'} l], [\tau_p k] \right\rangle.$$

Definition 1. Let G be a locally compact group and $1 < p < \infty$. We denote by $\mathcal{A}_p(G)$ the set of all pairs $\big((k_n), (l_n)\big)$ where (k_n) is a sequence of $\mathcal{L}^p(G)$ and (l_n) a sequence of $\mathcal{L}^{p'}(G)$with $\sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) < \infty$.

Let $\big((k_n), (l_n)\big)$ be an element of $\mathcal{A}_p(G)$. Then

$$\sum_{n=1}^{\infty} \|\overline{k}_n * \check{l}_n\|_u \le \sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n).$$

Therefore $\sum_{n=1}^{\infty} \left|\overline{k}_n * \check{l}_n\right|$ converges uniformly on G and $\sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n \in C_0(G)$.

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Definition 2. Let G be a locally compact group and $1 < p < \infty$. We denote by $A_p(G)$ the set of all $u \in \mathbb{C}^G$ such that there is $\Big((k_n), (l_n)\Big) \in \mathcal{A}_p(G)$ with

$$u(x) = \sum_{n=1}^{\infty} (\overline{k}_n * \check{l}_n)(x)$$

for every $x \in G$.

Proposition 1. *Let G be a locally compact group, $1 < p < \infty$, K a compact subset of G and U an open subset of G with $K \subset U$. Then there is $u \in A_p(G) \cap C_{00}(G)$ with:*

1. $0 \leq u(x) \leq 1$ *for every* $x \in G$,
2. $u(x) = 1$ *for every* $x \in K$,
3. $\operatorname{supp} u \subset U$.

Proof. There is V compact neighborhood of e in G such that $VV^{-1}K \subset U$. It suffices to put

$$u = k * \check{l} \quad \text{with} \quad k = \frac{1_V}{m(V)^{1/p}} \quad \text{and} \quad l = \frac{1_{K^{-1}V}}{m(V)^{1/p'}}.$$

Proposition 2. *Let G be a locally compact group and $1 < p < \infty$. $A_p(G)$ is a linear subspace of $C_0(G)$.*

Proof. It is clear that for $u \in A_p(G)$ and $\alpha \in \mathbb{C}$ the function αu belongs to $A_p(G)$. We claim that $u + v \in A_p(G)$ for $u, v \in A_p(G)$. Let $u = \sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n$ and $v = \sum_{n=1}^{\infty} \overline{r}_n * \check{s}_n$. For $n \in \mathbb{N}$ we put $f_{2n} = k_n$, $f_{2n-1} = r_n$, $g_{2n} = l_n$, $g_{2n-1} = s_n$. Then clearly $\Big((f_n), (g_n)\Big) \in \mathcal{A}_p(G)$ and $\sum_{n=1}^{\infty} \overline{f}_n * \check{g}_n = u + v$.

Definition 3. Let G be a locally compact group and $1 < p < \infty$. For every $u \in A_p(G)$ we put

$$\|u\|_{A_p} = \inf \left\{ \sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) \,\middle|\, \Big((k_n), (l_n)\Big) \in \mathcal{A}_p(G), \; u = \sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n \right\}.$$

Lemma 3. *Let E be a $\mathbb{C}$-vector space, p a seminorm on E and $\big(u_n\big)_{n=1}^{\infty}$ a Cauchy sequence of E. There is a subsequence $\big(u_{n_k}\big)_{k=1}^{\infty}$ with*

$$\sum_{k=1}^{\infty} p(u_{n_{k+1}} - u_{n_k}) < \infty.$$

Proof. For every $\varepsilon > 0$ there is $\lambda(\varepsilon) \in \mathbb{N}$ such that

$$p(u_n - u_m) < \varepsilon$$

for every $m, n \in \mathbb{N}$ with $m, n \geqslant \lambda(\varepsilon)$. By induction there is a sequence (n_k) of $\mathbb{N}$ with $n_k < n_{k+1}$ and

$$n_k \geqslant \lambda\left(\frac{1}{2^k}\right)$$

for every $k \in \mathbb{N}$. Then $(u_{n_k})_{k=1}^{\infty}$ is a subsequence of the sequence $(u_n)_{n=1}^{\infty}$. For every $k \in \mathbb{N}$, we have

$$p(u_{n_{k+1}} - u_{n_k}) < \frac{1}{2^k}$$

and therefore

$$\sum_{k=1}^{\infty} p(u_{n_{k+1}} - u_{n_k}) < \infty.$$

Theorem 4. *Let G be a locally compact group and $1 < p < \infty$. Then $\| \|_{A_p(G)}$ is a norm on $A_p(G)$ and with respect to this norm, $A_p(G)$ is a Banach space. For every $v \in A_p(G)$ we have $\|v\|_u \leqslant \|v\|_{A_p}$.*

Proof. The estimate $\|v\|_u \leqslant \|v\|_{A_p}$ is straightforward. The fact that $\| \|_{A_p}$ is a norm is consequence of the proof of Proposition 2. It remains to prove that $A_p(G)$ is complete. Let (v_n) be a Cauchy sequence of $A_p(G)$. The space $C_0(G)$ being complete, there is $v \in C_0(G)$ such that $\lim \|v - v_n\|_u = 0$. By Lemma 3 there is a subsequence (v_{n_j}) such that

$$\sum_{j=1}^{\infty} \|v_{n_{j+1}} - v_{n_j}\|_{A_p} < \infty.$$

For every $j \in \mathbb{N}$

$$v_{n_{j+1}} - v_{n_j} = \sum_{m=1}^{\infty} \overline{k}_{(m,j)} * \check{l}_{(m,j)}$$

and

$$\sum_{m=1}^{\infty} N_p(k_{(m,j)} N_{p'}(l_{(m,j)})) < \|v_{n_{j+1}} - v_{n_j}\|_{A_p} + \frac{1}{2^j}.$$

Then $\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\overline{k}_{(m,j)} * \check{l}_{(m,j)} \in A_p(G)$. Let $w = \sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\overline{k}_{(m,j)} * \check{l}_{(m,j)}$. We have

$$\lim_{t\to\infty}\left\| w - \sum_{j=1}^{t}\sum_{m=1}^{\infty}\overline{k}_{(m,j)} * \check{l}_{(m,j)}\right\|_{A_p} = 0.$$

But

$$\sum_{j=1}^{t}\sum_{m=1}^{\infty}\overline{k}_{(m,j)} * \check{l}_{(m,j)} = -v_{n_1} + v_{n_{t+1}}$$

and consequently

$$\lim_{t\to\infty}\|w + v_{n_1} - v_{n_{t+1}}\|_{A_p} = 0.$$

This implies $v = w + v_{n_1}$ and therefore $v \in A_p(G)$. We also obtain

$$\lim_{t\to\infty}\|v - v_{n_{t+1}}\|_{A_p} = 0$$

and finally

$$\lim_{n\to\infty}\|v - v_n\|_{A_p} = 0.$$

Remarks. 1. The Banach space $A_p(G)$ has been first considered by A. Figà-Talamanca in 1965 [44] for G abelian, for G compact but non necessarily commutative and also for G unimodular non-commutative, non-compact and $p = 2$. The above definition is due to Eymard [42].
2. For $u \in A_p(G)$ one has $\overline{u} \in A_p(G)$ and $\|\overline{u}\|_{A_p} = \|u\|_{A_p}$.
3. Every bounded measure μ on G belongs to the dual of $A_p(G)$ and $\|\mu\|_{A_p'} \le \|\mu\|$.

Lemma 5. *Let E be a $\mathbb{C}$-vector space, p a seminorm on E, F a $\mathbb{C}$-subspace of E, x in the closure of F in E and $\varepsilon > 0$. Then there is a sequence (y_n) in F such that:*

1. $\lim_{n\to\infty} p\Big(x - \sum_{k=1}^{n} y_k\Big) = 0,$
2. $\sum_{n=1}^{\infty} p(y_n) < p(x) + \varepsilon.$

Proof. Let $0 < \varepsilon_1 < \min\{1, \varepsilon\}$. For every $n \in \mathbb{N}$ there is $v_n \in F$ with

$$p(x - v_n) < \frac{\varepsilon_1}{2^{n+2}}.$$

Let $y_1 = v_1$ and $y_n = v_n - v_{n-1}$ for $n \geqslant 2$. For every $n \in \mathbb{N}$ we have

$$p\Big(x - (y_1 + \cdots + y_n)\Big) < \frac{1}{2^{n+2}}.$$

Let $n \geqslant 2$. We have

$$\sum_{k=1}^{n} p(y_k) = p(y_1) + \sum_{k=2}^{n} p(v_k - v_{k-1}),$$

but

$$\sum_{k=2}^{n} p(v_k - v_{k-1}) < 3\varepsilon_1 \sum_{k=2}^{n} \frac{1}{2^{k+2}} < \frac{3\varepsilon_1}{4}$$

and thus

$$\sum_{k=1}^{\infty} p(y_k) \leq p(x) + \frac{7\varepsilon_1}{8}.$$

Proposition 6. *Let G be a locally compact group,* $1 < p < \infty$, $\mathcal{E}_1$ *a dense subspace of* $\mathcal{L}^p(G)$, $\mathcal{E}_2$ *a dense subspace of* $\mathcal{L}^{p'}(G)$, $u \in A_p(G)$, *and* $\varepsilon > 0$. *Then there exists* (a_n) *a sequence of* $\mathcal{E}_1$ *and* (b_n) *a sequence of* $\mathcal{E}_2$ *such that*

$$\sum_{n=1}^{\infty} N_p(a_n) N_{p'}(b_n) < \|u\|_{A_p} + \varepsilon$$

and with $\sum_{n=1}^{\infty} \overline{a}_n * \check{b}_n = u$.

Proof. 1. For $f \in \mathcal{L}^p(G)$, $g \in \mathcal{L}^{p'}(G)$ and $\varepsilon > 0$ there exists (k_n) a sequence of $\mathcal{E}_1$ and (l_n) a sequence of $\mathcal{E}_2$ such that

$$\sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) < N_p(f) N_{p'}(g) + \varepsilon$$

and $\sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n = \overline{f} * \check{g}$.

Let $0 < \varepsilon_1 < \min\{1, \varepsilon\}$ and

$$0 < \eta < \frac{\varepsilon_1}{1 + N_p(f) + N_{p'}(g)} \ .$$

There is a (r_n) a sequence of $\mathcal{E}_1$ and (s_n) a sequence of $\mathcal{E}_2$ such that:

$$\lim N_p\left(f - \sum_{k=1}^{n} r_k\right) = 0, \sum_{n=1}^{\infty} N_p(r_n) < N_p(f) + \eta,$$

$$\lim N_{p'}\left(g - \sum_{k=1}^{n} s_k\right) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} N_{p'}(s_n) < N_{p'}(g) + \eta.$$

We have

$$\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty} N_p(r_n)N_{p'}(s'_n) < N_p(f)N_{p'}(g) + \varepsilon$$

and $\overline{f} * \check{g} = \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty} \overline{r}_n * \check{s}_{n'}$.

2. End of the proof.

Let $\Big((k_n), (l_n)\Big) \in \mathcal{A}_p(G)$ with $\sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n = u$ and

$$\sum_{n=1}^{\infty} N_p(k_n)N_{p'}(l_n) < \|u\|_{A_p} + \frac{\varepsilon}{2}.$$

By part (1), for every $n \in \mathbb{N}$ there is $\big(r_{(m,n)}\big)$ a sequence of $\mathcal{E}_1$ and $(s_{(m,n)})$ a sequence of $\mathcal{E}_2$ such that

$$\sum_{m=1}^{\infty} N_p(r_{(m,n)})N_{p'}(s_{(m,n)}) < N_p(k_n)N_{p'}(l_n) + \frac{\varepsilon}{2^{n+1}}$$

and $\overline{k}_n * \check{l}_n = \sum_{m=1}^{\infty} \overline{r}_{(m,n)} * \check{s}_{(m,n)}$. Finally

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} N_p(r_{(m,n)})N_{p'}(s_{(m,n)}) < \|u\|_{A_p} + \varepsilon$$

with $u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \overline{r}_{(m,n)} * \check{s}_{(m,n)}$.

Corollary 7. *Let G be a locally compact group and $1 < p < \infty$. Then $A_p(G) \cap C_{00}(G)$ is dense in $A_p(G)$.*

3.2 $A_2(G)$ for G Abelian

We recall that for G be a locally compact abelian group, $\Phi_{\hat{G}}$ (Definition 1 of Sect. 1.3) is an involutive contractive monomorphism of the Banach algebra $L^1(\widehat{G})$ into the Banach algebra $C_0(G)$.

Lemma 1. *Let G be a locally compact abelian group and let $\big((k_n),(l_n)\big)$ be an element of $\mathcal{A}_2(G)$. Then there is a unique $f \in L^1(\widehat{G})$ with*

$$\lim \left\| f - \sum_{n=1}^{N} \mathcal{F}(k_n)\tilde{}\,\mathcal{F}(l_n)\check{}\right\|_1 = 0.$$

Moreover we have:

1. $\Phi_{\hat{G}}(f) = \sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n,$
2. $\|f\|_1 \leqslant \sum_{n=1}^{\infty} N_2(k_n)N_2(l_n).$

Proof. Let $M, N \in \mathbb{N}$ with $M < N$. We have

$$\sum_{n=M}^{N} \left\|\mathcal{F}(k_n)\tilde{}\,\mathcal{F}(l_n)\check{}\right\|_1 \leq \sum_{n=M}^{N} N_2(k_n)N_2(l_n).$$

Consequently

$$\sum_{n=1}^{N} \mathcal{F}(k_n)\tilde{}\,\mathcal{F}(l_n)\check{},\; N = 1,2,3,\ldots,$$

is a Cauchy sequence in $L^1(\widehat{G})$. There is therefore $f \in L^1(\widehat{G})$ with

$$\lim \left\| f - \sum_{n=1}^{N} \mathcal{F}(k_n)\tilde{}\,\mathcal{F}(l_n)\check{}\right\|_1 = 0.$$

Let $n \in \mathbb{N}$ we have

$$\left\| \sum_{j=1}^{\infty} \overline{k}_j * \check{l}_j - \Phi_{\hat{G}}(f) \right\|_u \leqslant \sum_{j=n+1}^{\infty} N_2(k_j)N_2(l_j) + \left\| f - \sum_{j=1}^{n} \mathcal{F}(k_j)\tilde{}\,\mathcal{F}(l_j)\check{}\right\|_1$$

and

$$\|f\|_1 \leqslant \left\| f - \sum_{j=1}^{n} \mathcal{F}(k_j)\tilde{}\,\mathcal{F}(l_j)\check{}\right\|_1 + \sum_{j=1}^{\infty} N_2(k_j)N_2(l_j).$$

Theorem 2. *Let G be a locally compact abelian group. Then $A_2(G)$ is an involutive Banach algebra for the complex conjugation and the pointwise product. The map $\Phi_{\hat{G}}$ is an involutive isometric isomorphism of the Banach algebra $L^1(\widehat{G})$ onto $A_2(G)$. For every $u \in A_2(G)$ there is $k, l \in \mathcal{L}^2(G)$ with $u = \overline{k} * \check{l}$ and $\|u\|_{A_2} = N_2(k)N_2(l)$.*

Proof. 1. $\Phi_{\hat{G}}$ is an isometry of $L^1(\widehat{G})$ onto $A_2(G)$.

Let $f \in L^1(\widehat{G})$ and $f_1 \in f$. We define $r = |f_1|^{1/2}$ and $s(\chi) = 0$ if $f_1(\chi) = 0$ and $s(\chi) = \dfrac{f_1(\chi)}{|f_1(\chi)|^{1/2}}$ if $f_1(\chi) \neq 0$. Then

$$\Phi_{\hat{G}}(f) = \overline{\mathcal{F}^{-1}\big([\tilde{r}]\big)} * \mathcal{F}^{-1}\big([\check{s}]\big)^{\check{}}$$

and therefore $\Phi_{\hat{G}}(f) \in A_2(G)$ with

$$\|\Phi_{\hat{G}}(f)\|_{A_2} \leqslant N_2(\tilde{r})N_2(\check{s}) = N_1(f_1) = \|f\|_1.$$

Let any $\big((k_n), (l_n)\big) \in \mathcal{A}_2(G)$ with $\Phi_{\hat{G}}(f) = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n$. There is $g \in L^1(\widehat{G})$ with $\Phi_{\hat{G}}(g) = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n$ and

$$\|g\|_1 \leqslant \sum_{n=1}^{\infty} N_2(k_n)N_2(l_n).$$

This implies $g = f$ and $\|f\|_1 = \|\Phi_{\hat{G}}(f)\|_{A_2}$.

Let $u \in A_2(G)$. Choose $\big((k_n), (l_n)\big) \in \mathcal{A}_2(G)$ with $u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n$ and

$$f = \sum_{n=1}^{\infty} \mathcal{F}(k_n)^{\tilde{}} \mathcal{F}(l_n)^{\check{}}.$$

Then $\Phi_{\hat{G}}(f) = u$.

2. For every $u \in A_2(G)$ there is $k, l \in \mathcal{L}^2(G)$ with $u = \overline{k} * \check{l}$ and $\|u\|_{A_2} = N_2(k) N_2(l)$.

Let $f \in \Phi_{\hat{G}}^{-1}(u)$. Consider $r = |f|^{1/2}$ and $s(\chi) = 0$ if $f(\chi) = 0$, $s(\chi) = \dfrac{f(\chi)}{|f(\chi)|^{1/2}}$ if $f(\chi) \neq 0$. It suffices to choose $k \in \mathcal{F}^{-1}\big([\tilde{r}]\big)$ and $l \in \mathcal{F}^{-1}\big([\check{s}]\big)$.

3. $\Phi_{\hat{G}}(f * g) = \Phi_{\hat{G}}(f)\Phi_{\hat{G}}(g)$ and $\Phi_{\hat{G}}(\tilde{f}) = \overline{\Phi_{\hat{G}}(f)}$ for $f, g \in L^1(\widehat{G})$.

We have $(f * g)\widehat{\ } = \widehat{f}\widehat{g}$ and $(\tilde{f})\widehat{\ } = \overline{\widehat{f}}$.

Remark. For $u \in A_2(G)$ we have $\check{u} \in A_2(G)$ and $\|\check{u}\|_{A_2} = \|u\|_{A_2}$.

3.3 $A_p(G)$ is a Banach Algebra

Let μ be a complex Radon measure on the locally compact Hausdorff space X, and $(V, \| \|_V)$ a complex normed space. For $f : X \to V$ and $1 \leq p < \infty$ we put

$$N_p(f) = \left(\int_X^* \|f(x)\|_V^p \, d|\mu|(x) \right)^{1/p}.$$

Let $\mathcal{F}_V^p(X, \mu)$ be the $\mathbb{C}$-subspace of all maps of X in V with $N_p(f) < \infty$. Then N_p is a semi-norm on $\mathcal{F}_V^p(X, \mu)$. This semi-normed space is complete. We denote by $\mathcal{L}_V^p(X, \mu)$ the closure of $C_{00}(X; V)$ into $\mathcal{F}^p(X, \mu; V)$ and by $L_V^p(X, \mu)$ the associated normed space. For $f \in \mathbb{C}^X$ and $v \in V$ we define $fv \in V^X$ by $(fv)(x) = f(x)v$. Let $\mathcal{A}(X; V)$ be the linear span of $\{\varphi v | \varphi \in C_{00}(X), v \in V\}$ in V^X and μ the canonical linear extension of μ to $\mathcal{A}(X; V)$. For every $f \in \mathcal{A}(X; V)$ we have $\|\mu(f)\|_V \leq N_1(f)$.

Lemma 1. *Let X be a locally compact Hausdorff space, μ a complex Radon measure on X and $(V, \| \|_V)$ a complex normed space. Then $\mathcal{A}(X; V)$ is dense in $\mathcal{L}_V^1(X, \mu)$.*

Proof. Let $f \in \mathcal{L}_V^1(X, \mu)$ and $\varepsilon > 0$. There is $f_1 \in C_{00}(X; V)$ with

$$N_1(f - f_1) < \frac{\varepsilon}{2}.$$

There is also $g \in C_{00}(X)$ with $g \geqslant 0$ and $g(x) = 1$ on supp f_1. There is $O_1, \ldots, O_N$ open subsets of X, $a_1 \in O_1, \ldots, a_N \in O_N$ with supp $f_1 \subset O_1 \cup \cdots \cup O_N$ and

$$\|f_1(x) - f_1(a_n)\|_V < \frac{\varepsilon}{2(1 + |\mu|(g))}$$

for every $x \in O_n$ and for every $1 \leqslant n \leqslant N$. We can find $e_1, \ldots, e_N \in C_{00}(X)$ with $e_n \geqslant 0$, supp $e_n \subset O_n$ for every $1 \leqslant n \leqslant N$, $\sum_{n=1}^{N} e_n(x) = 1$ on supp f_1 and $\sum_{n=1}^{N} e_n(x) \leqslant 1$ on X. Then $k = \sum_{n=1}^{N} g\, e_n f_1(a_n)$ belongs to the vectorspace $\mathcal{A}(X; V)$. For every $x \in X$ we have

$$f_1(x) - k(x) = \sum_{n=1}^{N} g(x) e_n(x) \big(f_1(x) - f_1(a_n)\big).$$

This implies $N_1(f - k) < \varepsilon$.

Theorem 2. *Let X be a locally compact Hausdorff space, μ a complex Radon measure on X and $(V, \| \|_V)$ a complex Banach space and $f \in \mathcal{L}_V^1(X, \mu)$. Then* *there is a unique* $v \in V$ *such that* $F(v) = \int_X F(f(x))d\mu(x)$ *for every* $F \in V'$. *We have* $\|v\|_V \le N_1(f)$.

Proof. There is a sequence (f_n) of $\mathcal{A}(X;V)$ with $\lim_{n\to\infty} N_1(f_n - f) = 0$. For $m, n \in \mathbb{N}$, we have $\|\mu(f_m) - \mu(f_n)\|_V \leqslant N_1(f_m - f_n)$. The sequence $(\mu(f_n))$ is a Cauchy sequence in the Banach space V. Then there is $v \in V$ with $\lim \|\mu(f_n) - v\|_V = 0$. Let $F \in V'$. For every $n \in \mathbb{N}$ we have

$$\left|F(v) - \int_X F(f(x))d\mu(x)\right| \leqslant \|F\|_{V'} \, \|v - \mu(f_n)\|_V + \|F\|_{V'} \, N_1(f_n - f).$$

It follows that $F(v) = \int_X F(f(x))d\mu(x)$. The uniqueness of v is straightforward.

Definition 1. The vector v of Theorem 2, is denoted $\mu(f)$ or $\int_X f(x)d\mu(x)$.

Remarks. 1. μ is a continuous linear map of $\mathcal{L}^1_V(X,\mu)$ into V.
2. See Bourbaki, [6], Chap. III, Sect. 3, no. 3, Corollaire 2, p. 80 for a more general result.

Let ω be a continuous map of a topological space X in a topological space Y. For $f \in C(Y)$ we put $\omega^*(f) = f \circ \omega$. Then ω^* is an algebra homomorphism of $C(Y)$ into $C(X)$.

Lemma 3. *Let* ω *be a continuous homomorphism of a locally compact group* G *into a locally compact group* H, $k, l \in \mathcal{M}^\infty_{00}(G)$ *and* $r, s \in C_{00}(H)$. *We set for* $h \in H$,

$$j(h) = \overline{k\ \omega^*(r_h)} *_G (l\ (\omega^*(s_h))\check{}\,.$$

Then for $1 < p < \infty$ *we have:*

1. $j \in C_{00}(H; A_p(G))$ *and* $\int_H j(h)dh \in A_p(G)$,
2. $\int_H j(h)dh = \overline{k} *_G \check{l}\omega^*(\overline{r} *_H \check{s})$,
3. $\|\overline{k} *_G \check{l}\ \omega^*(\overline{r} *_H \check{s})\|_{A_p} \leqslant N_p(k)N_{p'}(l)N_p(r)N_{p'}(s)$.

Proof. For every $h \in H$ we put $f(h) = k\ \omega^*(r_h)$ and $g(h) = l\ \omega^*(s_h)$. Then $f(h), g(h) \in \mathcal{M}^\infty_{00}(G)$ and $j(h) = \overline{f(h)} *_G g(h)\check{}$. But for $h, h_0 \in H$ we have

$$\|j(h) - j(h_0)\|_{A_p}$$
$$\leqslant \|r\|_u\ N_p(k)\ N_{p'}\Big(g(h) - g(h_0)\Big) + \|r_h - r_{h_0}\|_u\ N_p(k)\ N_{p'}\Big(g(h_0)\Big).$$

This implies $j \in C_{00}(H; A_p(G))$, $\int\limits_H j(h)dh \in A_p(G)$ and

$$\left\| \int\limits_H j(h)dh \right\|_{A_p} \leq \int\limits_G \|j(h)\|_{A_p} dh.$$

But

$$\int\limits_H \|j(h)\|_{A_p} dh \leqslant \int\limits_H N_p(f(h))\, N_{p'}(g(h))dh \leqslant \left(\int\limits_H N_p(f(h))^p dh\right)^{1/p} \left(\int\limits_H N_{p'}(g(h))^{p'} dh\right)^{1/p'}.$$

We have moreover

$$\left(\int\limits_H N_p(f(h))^p dh\right)^{1/p} = N_p(k)N_p(r) \quad \text{and} \quad \left(\int\limits_H N_{p'}(g(h))^{p'} dh\right)^{1/p'} = N_{p'}(l)N_{p'}(s)$$

and thus

$$\left\| \int\limits_H j(h)dh \right\|_{A_p} \leqslant N_p(k)N_{p'}(l)N_p(r)N_{p'}(s).$$

Let x be an element of G. We recall that $\delta_x \in A_p(G)'$, thus

$$\delta_x\left(\int\limits_H j(h)dh\right) = \int\limits_H j(h)(x)dh.$$

But for every $h \in H$

$$j(h)(x) = \int\limits_G \overline{k(xt)}\ \overline{r(\omega(xt)h)} l(t)s(\omega(t)h)dt$$

and therefore

$$\int\limits_H j(h)(x)dh = (\overline{k} *_G \check{l})(x)(\overline{r} *_H \check{s})(\omega(x)).$$

Theorem 4. *Let ω be a continuous homomorphism of a locally compact group G into a locally compact group H and $1 < p < \infty$. For $u \in A_p(G)$ and $v \in A_p(H)$ we have then $u\,\omega^*(v) \in A_p(G)$ and $\|u\,\omega^*(v)\|_{A_p} \leqslant \|u\|_{A_p}\|v\|_{A_p}$.*

Proof. Let $\varepsilon > 0$. We choose

$$0 < \varepsilon_1 < \min\left\{1, \frac{\varepsilon}{\left(1 + \|u\|_{A_p} + \|v\|_{A_p}\right)}\right\}.$$

By Proposition 6 of Sect. 3.1 there is (k_n), (l_n), (r_n) and (s_n) sequences of $C_{00}(G)$ with

$$u = \sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n, \sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) < \|u\|_{A_p} + \varepsilon_1, v = \sum_{n=1}^{\infty} \overline{r}_n * \check{s}_n$$

and

$$\sum_{n=1}^{\infty} N_p(r_n) N_{p'}(s_n) < \|v\|_{A_p} + \varepsilon_1.$$

For every $n \in \mathbb{N}$ we put $u_n = \sum_{j=1}^{n} \overline{k}_j * \check{l}_j$ and $v_n = \sum_{j=1}^{n} \overline{r}_j *_H \check{s}_j$. By Lemma 3 $u_n\, \omega^*(v_n) \in A_p(G)$. For $n > m \geqslant 1$ we have

$$\|u_n\, \omega^*(v_n) - u_m\, \omega^*(v_m)\|_{A_p} \leqslant \big(\|v\|_{A_p} + 1\big)\left(\sum_{j=m+1}^{\infty} N_p(k_j) N_{p'}(l_j) \right) + \big(\|u\|_{A_p} + 1\big)\left(\sum_{i=m+1}^{\infty} N_p(r_i) N_{p'}(s_i) \right).$$

There is consequently $w \in A_p(G)$ with

$$\lim \|w - u_n\, \omega^*(v_n)\|_{A_p(G)} = 0.$$

For every $n \in \mathbb{N}$ we have

$$\|w - u\,\omega^*(v)\|_\infty \leqslant \big(1 + \|u\|_{A_p}\big)\left(\sum_{j=n+1}^{\infty} N_p(r_j) N_{p'}(s_j) \right) + \|v\|_\infty \left(\sum_{j=1+n}^{\infty} N_p(k_j) N_{p'}(l_j) \right)$$

and therefore $u\,\omega^*(v) \in A_p(G)$. We finally have

$$\|u\,\omega^*(v)\|_{A_p} \leqslant \|u\,\omega^*(v) - u_n\, \omega^*(v_n)\|_{A_p} + \|u\|_{A_p}\, \|v\|_{A_p} + \varepsilon.$$

Corollary 5. *Let G be a locally compact group and $1 < p < \infty$. For the pointwise product the Banach space $A_p(G)$ is a commutative Banach algebra.*

Remarks. 1. Even for $G = \mathbb{T}$ this result is not trivial.

2. The fact that $A_p(G)$ is a Banach algebra is due Herz in ([56], p. 6002, Théorème 1, [57], p. 244, [59], p. 72, Corollary). Eymard proved earlier that $A_2(G)$ is a Banach algebra [41]. The proof above is due to Spector ([112], Lemme IV.2.1, p. 52, and Théorème IV.2.3., p. 54). See the notes to Chap. 3 for Herz' approach and various generalizations.

Corollary 6. *Let G be a locally compact group and $1 < p < \infty$. Let $v \in A_p(G)$, for every $u \in A_p(G_d)$ we have $uv \in A_p(G_d)$ and $\|vu\|_{A_p(G_d)} \leq \|u\|_{A_p(G_d)} \|v\|_{A_p(G)}$.*

Chapter 4
The Dual of $A_p(G)$

The dual of the Banach space $A_p(G)$ is the Banach space $PM_p(G)$ of all limits of convolution operators associated to bounded measures. If G is abelian then $A_2(G) \subset A_p(G)$. Holomorphic functions operate on $A_p(G)$.

4.1 The Dual of $A_p(G)$: The Notion of Pseudomeasure

Proposition 1. *Let G be an abelian locally compact group. Then for $u \in A_2(G)$ and $\mu \in M^1(G)$ we have*

$$\mu(u) = \left\langle \Phi_{\hat{G}}^{-1}(u), \left(\widehat{\overline{\mu}}\right)^{\cdot} \right\rangle$$

and

$$|\mu(u)| \leq \|\widehat{\mu}\|_\infty \|u\|_{A_2}.$$

Proof. We have

$$\mu(u) = \int_G \left(\int_{\hat{G}} \Phi_{\hat{G}}^{-1}(u)(\chi)\varepsilon_G(x)(\chi)d\chi \right) d\mu(x) = \int_{\hat{G}} \Phi_{\hat{G}}^{-1}(u)(\chi) \left(\int_G \chi(x)d\mu(x) \right) d\chi.$$

Moreover

$$|\mu(u)| \leq \|\widehat{\mu}\|_\infty \|\Phi_{\hat{G}}^{-1}(u)\|_1 = \|\widehat{\mu}\|_\infty \|u\|_{A_2}.$$

According to Theorem 2 of Sect. 1.3 $\left|\!\left|\!\left| \lambda_G^p(\tilde{\mu}) \right|\!\right|\!\right|_2 = \|\widehat{\mu}\|_\infty$. Proposition 2 below extends the inequality of Proposition 1 to every $1 < p < \infty$ and to every nonabelian locally compact group.

A. Derighetti, *Convolution Operators on Groups*, Lecture Notes of the Unione Matematica Italiana 11, DOI 10.1007/978-3-642-20656-6_4,

Proposition 2. *Let G be a locally compact group and $1 < p < \infty$. Then for $\mu \in M^1(G)$ and $u \in A_p(G)$ we have*

$$\mu(u) = \sum_{n=1}^{\infty} \overline{\left\langle \lambda_G^p(\tilde{\mu})[\tau_p k_n], [\tau_{p'} l_n] \right\rangle}$$

for every $\big((k_n), (l_n)\big) \in \mathcal{A}_p(G)$ with $u = \sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n$. The following inequality holds

$$|\mu(u)| \le \left\|\left| \lambda_G^p(\tilde{\mu}) \right|\right\|_p \|u\|_{A_p}.$$

Proof. Let $\varphi, \psi \in C_{00}(G)$. For $x \in G$ we have

$$(\overline{\varphi} * \check{\psi})(x) = \int_G \overline{(\tau_p \varphi)_{x^{-1}}(y)} \Delta_G(x^{-1})^{1/p} (\tau_{p'} \psi)(y) dy$$

and therefore

$$\mu(\overline{\varphi} * \check{\psi}) = \int_G (\tau_{p'} \psi)(y) \left(\int_G \overline{(\tau_p \varphi)_{x^{-1}}(y)} \Delta_G(x^{-1})^{1/p} d\mu(x) \right) dy,$$

but

$$\int_G \overline{(\tau_p \varphi)_{x^{-1}}(y)} \Delta_G(x^{-1})^{1/p} d\mu(x) = \overline{(\tau_p \varphi * \Delta_G^{1/p'} \tilde{\mu})(y)}$$

hence

$$\mu(\overline{\varphi} * \check{\psi}) = \overline{\left\langle \lambda_G^p(\tilde{\mu})[\tau_p \varphi], [\tau_{p'} \psi] \right\rangle}.$$

For $k \in \mathcal{L}^p(G)$ and $l \in \mathcal{L}^{p'}(G)$ choose $\varphi_n, \psi_n \in C_{00}(G)$, $n = 1, 2, 3, \ldots$, such that $\varphi_n \to k$ in $\mathcal{L}^p(G)$ and $\psi_n \to l$ in $\mathcal{L}^{p'}(G)$. Then

$$\lim_{n \to \infty} \overline{\left\langle \lambda_G^p(\tilde{\mu})[\tau_p \varphi_n], [\tau_{p'} \psi_n] \right\rangle} = \overline{\left\langle \lambda_G^p(\tilde{\mu})[\tau_p k], [\tau_{p'} l] \right\rangle}.$$

On the other hand

$$\lim_{n \to \infty} \overline{\varphi_n} * \check{\psi}_n = \overline{k} * \check{l}$$

in $A_p(G)$ and thus

$$\overline{\left\langle \lambda_G^p(\tilde{\mu})[\tau_p k], [\tau_{p'} l] \right\rangle} = \mu(\overline{k} * \check{l}).$$

Finally we have

$$\mu(u) = \mu\left(\sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n \right) = \sum_{n=1}^{\infty} \mu(\overline{k_n} * \check{l}_n) = \sum_{n=1}^{\infty} \overline{\left\langle \lambda_G^p(\tilde{\mu})[\tau_p k_n], [\tau_{p'} l_n] \right\rangle}$$

and

$$|\mu(u)| \leq \sum_{n=1}^{\infty} \left| \left\langle \lambda_G^p(\tilde{\mu})[\tau_p k_n], [\tau_{p'} l_n] \right\rangle \right| \leq \left\| \lambda_G^p(\tilde{\mu}) \right\|_p \sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n).$$

Remark. From Theorem 2 of Sect. 1.3 and Theorem 2 of Sect. 3.2 it follows, for G abelian, that

$$\left\| \lambda_G^2(\tilde{\mu}) \right\|_2 = \sup \left\{ |\mu(u)| \middle| u \in A_2(G), \|u\|_{A_2} \leq 1 \right\}.$$

We will prove the following generalization:

$$\left\| \lambda_G^p(\tilde{\mu}) \right\|_p = \sup \left\{ |\mu(u)| \middle| u \in A_p(G), \ \|u\|_{A_p} \leq 1 \right\}$$

for every $1 < p < \infty$ and for an arbitrary locally compact group G.

Scholium 3. *Let G be a locally compact group and $1 < p < \infty$. Then for every $\mu \in M^1(G)$ we have $\lambda_G^p(\mu) = \int_G \lambda_G^p(\delta_x) d\mu(x)$ in the following sense: for every $\varphi \in \mathcal{L}^p(G)$ and every $\psi \in \mathcal{L}^{p'}(G)$*

$$\left\langle \lambda_G^p(\mu)[\varphi], [\psi] \right\rangle = \int_G \left\langle \lambda_G^p(\delta_x)[\varphi], [\psi] \right\rangle d\mu(x).$$

This Scholium permits to complete in a very simple way Proposition 4 of Sect. 1.2.

Corollary 4. *Let G be a locally compact group and $1 < p < \infty$. Then λ_G^p is a contractive representation of the Banach algebra $M^1(G)$ into $L^p(G)$.*

Proof. It suffices to verify that $\lambda_G^p(\alpha * \beta) = \lambda_G^p(\alpha) \circ \lambda_G^p(\beta)$ for $\alpha, \beta \in M^1(G)$. Let $\varphi \in L^p(G)$ and $\psi \in L^{p'}(G)$. We have

$$\begin{aligned}
\left\langle \lambda_G^p(\alpha * \beta)\varphi, \psi \right\rangle &= \int_G \left\langle \lambda_G^p(\delta_x)\varphi, \psi \right\rangle d(\alpha * \beta)(x) \\
&= \int_G \left(\int_G \left\langle \lambda_G^p(\delta_{xy})\varphi, \psi \right\rangle d\alpha(x) \right) d\beta(y) \\
&= \int_G \left(\int_G \left\langle \lambda_G^p(\delta_x)\lambda_G^p(\delta_y)\varphi, \psi \right\rangle_{L^p, L^{p'}} d\alpha(x) \right) d\beta(y) \\
&= \int_G \left\langle \lambda_G^p(\alpha)\lambda_G^p(\delta_y)\varphi, \psi \right\rangle d\beta(y)
\end{aligned}$$

$$= \int_G \langle \lambda_G^p(\delta_y)\varphi, \lambda_G^{p'}(\alpha)^*\psi \rangle d\beta(y) = \langle \lambda_G^p(\beta)\varphi, \lambda_G^{p'}(\alpha)^*\psi \rangle$$
$$= \langle \lambda_G^p(\alpha)\lambda_G^p(\beta)\varphi, \psi \rangle.$$

Definition 1. Let G be a locally compact group and $1 < p < \infty$. The topology on $\mathcal{L}(L^p(G))$, associated to the family of seminorms

$$T \mapsto \left| \sum_{n=1}^{\infty} \langle T[k_n], [l_n] \rangle \right|$$

with $\left((k_n), (l_n)\right) \in \mathcal{A}_p(G)$, is called the ultraweak topology.

Remarks. 1. This topology is locally convex and Hausdorff.
2. Let T be a continuous operator of $L^p(G)$, (S_α) a net of $\mathcal{L}(L^p(G))$ such that $\sup_\alpha |||S_\alpha|||_p < \infty$, $\mathcal{E}$ a dense subset of $L^p(G)$ and $\mathcal{F}$ a dense subset of $L^{p'}(G)$. Suppose that
$$\lim_\alpha < S_\alpha\varphi, \psi > = < T\varphi, \psi >$$
for $\varphi \in \mathcal{E}$ and $\psi \in \mathcal{F}$. Then $\lim S_\alpha = T$ for the ultraweak topology.

Definition 2. Let G be a locally compact group and $1 < p < \infty$. The closure of $\lambda_G^p(M^1(G))$ in $\mathcal{L}(L^p(G))$ with respect to the ultraweak topology is denoted $PM_p(G)$. Every element of $PM_p(G)$ is called a p-pseudomeasure.

Remarks. 1. Clearly $PM_p(G) \subset CV_p(G)$.
2. Theorem 4 of Sect. 2.3 implies that $CV_2(G) = PM_2(G)$ for every locally compact unimodular group.
3. We will show that $PM_p(G) = CV_p(G)$ for every locally compact amenable group G and every $1 < p < \infty$.
4. It is unknown whether $PM_p(G) = CV_p(G)$ in general.

Lemma 5. *Let G be a locally compact group, $1 < p < \infty$, $T \in PM_p(G)$, $\left((k_n), (l_n)\right)$ and $\left((k_n'), (l_n')\right) \in \mathcal{A}_p(G)$ with $\sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n = \sum_{n=1}^{\infty} \overline{k_n'} * \check{l}_n'$. Then*

$$\sum_{n=1}^{\infty} \left\langle T[\tau_p k_n], [\tau_{p'} l_n] \right\rangle = \sum_{n=1}^{\infty} \left\langle T[\tau_p k_n'], [\tau_{p'} l_n'] \right\rangle.$$

Proof. Let $\varepsilon > 0$. There is $\mu \in M^1(G)$ such that

$$\left| \sum_{n=1}^{\infty} \left\langle T[\tau_p k_n], [\tau_{p'} l_n] \right\rangle - \sum_{n=1}^{\infty} \left\langle \lambda_G^p(\mu)[\tau_p k_n], [\tau_{p'} l_n] \right\rangle \right| < \frac{\varepsilon}{2}$$

and

$$\left|\sum_{n=1}^{\infty}\left\langle T[\tau_p k_n'],[\tau_{p'} l_n']\right\rangle - \sum_{n=1}^{\infty}\left\langle \lambda_G^p(\mu)[\tau_p k_n'],[\tau_{p'} l_n']\right\rangle\right| < \frac{\varepsilon}{2}.$$

Proposition 2 implies

$$\sum_{n=1}^{\infty}\left\langle \lambda_G^p(\mu)[\tau_p k_n],[\tau_{p'} l_n]\right\rangle = \sum_{n=1}^{\infty}\left\langle \lambda_G^p(\mu)[\tau_p k_n'],[\tau_{p'} l_n']\right\rangle$$

and consequently

$$\left|\sum_{n=1}^{\infty}\left\langle T[\tau_p k_n],[\tau_{p'} l_n]\right\rangle - \sum_{n=1}^{\infty}\left\langle T[\tau_p k_n'],[\tau_{p'} l_n']\right\rangle\right| < \varepsilon.$$

Definition 3. Let T be a p-pseudomeasure on the locally compact group G, and $1 < p < \infty$. We define the continuous linear form $\Psi_G^p(T)$ on $A_p(G)$ by

$$\Psi_G^p(T)(u) = \sum_{n=1}^{\infty}\overline{\left\langle T[\tau_p k_n],[\tau_{p'} l_n]\right\rangle},\ u = \sum_{n=1}^{\infty}\overline{k_n} * \check{l}_n \in A_p(G).$$

The following theorem gives a description of the dual of $A_p(G)$.

Theorem 6. *Let G be a locally compact group and* $1 < p < \infty$*. Then:*

1. Ψ_G^p *is a conjugate linear isometry of* $PM_p(G)$ *onto* $A_p(G)'$;
2. $\Psi_G^p(\lambda_G^p(\tilde{\mu})) = \mu$ *for every* $\mu \in M^1(G)$;
3. Ψ_G^p *is a homeomorphism of* $PM_p(G)$*, with the ultraweak topology, onto* $A_p(G)'$*, with the weak topology* $\sigma(A_p(G)', A_p(G))$.

Proof. (I) Let $F \in A_p(G)'$. We prove that there is a unique operator $T_F \in \mathcal{L}(L^p(G))$ such that

$$F(\overline{k} * \check{l}) = \overline{\left\langle T_F[\tau_p k],[\tau_{p'} l]\right\rangle}$$

for every $k \in \mathcal{L}^p(G)$ and every $l \in \mathcal{L}^{p'}(G)$ and that $T_F \in CV_p(G)$ with $|||T_F|||_p = \|F\|_{A_p(G)'}$.

The map $([k],[l]) \mapsto \overline{F(\overline{k} * \check{l})}$ is linear in the first variable and conjugate linear in the second variable and

$$\left|F(\overline{k} * \check{l})\right| \leqslant \|F\|_{A_p'}\|[k]\|_p\|[l]\|_{p'}.$$

Hence there is an unique operator $S_F \in \mathcal{L}(L^p(G))$ with

$$\overline{F(\overline{k} * \check{l})} = \left\langle S_F[k],[l]\right\rangle.$$

We have $|||S_F|||_p \leqslant \|F\|_{A'_p}$. It is straightforward to verify that $S_F \in CV_p^d(G)$. Let $T_F = \tau_p \circ S_F \circ \tau_p$. Then $T_F \in CV_p(G)$ and

$$F(\overline{k} * \check{l}) = \overline{\left\langle T_F[\tau_p k], [\tau_{p'} l]\right\rangle}$$

for every $k \in \mathcal{L}^p(G)$ and every $l \in \mathcal{L}^{p'}(G)$. It remains to prove that $\|F\|_{A'_p} \leqslant |||T_F|||_p$.

For every $u \in A_p(G)$ and every $\Big((k_n), (l_n)\Big) \in \mathcal{A}_p(G)$ with $u = \sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n$ we have

$$F(u) = \sum_{n=1}^{\infty} \overline{\left\langle T_F[\tau_p k_n], [\tau_{p'} l_n]\right\rangle}.$$

This implies

$$|F(u)| \leqslant |||T_F|||_p \sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n)$$

and therefore $\|F\|_{A'_p} \leq |||T_F|||_p$.

(II) For every $F \in A'_p$ we put $j(F) = T_F$. Then clearly:

i. $j(F_1 + F_2) = j(F_1) + j(F_2)$ for $F_1, F_2 \in A'_p$,
ii. $j(\alpha F) = \overline{\alpha} j(F)$ for $\alpha \in \mathbb{C}$ and $F \in A'_p$,
iii. $j(\mu) = \lambda_G^p(\tilde{\mu})$ for $\mu \in M^1(G)$,
iv. $|||j(F)|||_p = \|F\|_{A'_p}$ for $F \in A'_p$. In particular the map j is injective.

We verify only (*iii*). According to Proposition 2 we have

$$\mu\Big(\overline{k} * \check{l}\Big) = \overline{\left\langle \lambda_G^p(\tilde{\mu})[\tau_p k], [\tau_{p'} l]\right\rangle}$$

but

$$\mu\Big(\overline{k} * \check{l}\Big) = \overline{\left\langle T_\mu[\tau_p k], [\tau_{p'} l]\right\rangle}.$$

(III) The set $\Big\{f m_G \Big| f \in C_{00}(G)\Big\}$ is dense in $A_p(G)'$ with respect to the topology $\sigma(A'_p, A_p)$.

Suppose that E, F are two complex vector spaces and B a bilinear form on $E \times F$. For M a subset of E we put $M^0 = \Big\{u \in F \Big| B(L, u) = 0$ for every $L \in M\Big\}$ and similarly for a subset of F. We also set $M^{00} = (M^0)^0$. Suppose now that M is a subspace of E. By the bipolar theorem, the closure of M with respect to the topology $\sigma_B(E, F)$ coincides with M^{00}. Choose $E = A_p(G)'$, $F = A_p(G)$, $B(L, u) = L(u)$ and $M = \Big\{f m_G \Big| f \in C_{00}(G\Big\}$. If $u \in M^0$, then $f m_G(u) = 0$

for every $f \in C_{00}(G)$, and therefore $u = 0$. We have proved that $M^0 = 0$ and consequently that $M^{00} = E$.
(IV) Consider on $A_p(G)'$ the topology $\sigma(A'_p, A_p)$ and on $j(A_p(G)')$ the ultraweak topology. Then j is a homeomorphism of $A_p(G)'$ onto $j(A_p(G)')$.

Let $\varepsilon > 0$, $\left((k_n), (l_n)\right) \in \mathcal{A}_p(G)$ and $u = \sum_{n=1}^{\infty} \overline{\tau_p k_n} * \left(\tau_{p'} l_n\right)\check{}$. Let

$$U = \left\{L \in A_p(G)' \middle| |L(u)| < \varepsilon\right\}$$

and

$$V = \left\{T \in j\left(A_p(G)'\right) \middle| \left|\sum_{n=1}^{\infty} \left\langle T[k_n], [l_n]\right\rangle\right| < \varepsilon\right\}.$$

We have $j(U) = V$ and $U = j^{-1}(V)$.
(V) We have $j(A_p(G)') \subset PM_p(G)$.

Let $F \in A_p(G)'$. According to (III) there is a net (f_α) of $C_{00}(G)$ with $\lim f_\alpha m_G = F$ for $\sigma(A'_p, A_p)$ and therefore $\lim j(f_\alpha m_G) = j(F)$ for the ultraweak topology.
(VI) j is a bijection of $A_p(G)'$ onto $PM_p(G)$ and $\Psi_G^p = j^{-1}$.

It is straightforward to verify that for $T \in PM_p(G)$ $\Psi_G^p(T) \in A_p(G)'$ and $\|\Psi_G^p(T)\|_{A'_p} \le \|\!|T|\!\|_p$. But for $k \in \mathcal{L}^p(G)$ and $l \in \mathcal{L}^{p'}(G)$ we have

$$\Psi_G^p(T)\left(\overline{\tau_p k} * (\tau_{p'} l)\check{}\right) = \overline{\left\langle j\left(\Psi_G^p(T)\right)[k], [l]\right\rangle} = \overline{\left\langle T[k], [l]\right\rangle}$$

and therefore $j\left(\Psi_G^p(T)\right) = T$.

Remark. This result is due to Herz [56], [57]. Figà-Talamanca [44] (Theorem 1, p. 496) proved it for $p > 1$ for G abelian or compact, and for $p = 2$ if G is unimodular. For $p = 2$ and any locally compact group it was proved by Eymard [41] (p. 210 (3.10) Théorème).

Corollary 7. *Let G be a locally compact group and $1 < p < \infty$. Then:*

1. $\left\{\lambda_G^p(f) \middle| f \in C_{00}(G)\right\}$ *is ultraweakly dense in $PM_p(G)$,*
2. $\left\{\lambda_G^p(\mu) \middle| \mu \text{ is a finitely supported complex measure on } G\right\}$ *is ultraweakly dense in $PM_p(G)$.*

Proof. The statement (1) is contained in the preceding proof. The proof of (2) is identical to the proof of (1) with $M = \left\{\mu \middle| \mu \text{ is a finitely supported complex measure on } G\right\}$.

Remark. For every $\mu \in M^1(G)$ there is a net (μ_α) of finitely supported complex measures with $\lim \lambda_G^p(\mu_\alpha) = \lambda_G^p(\mu)$ ultraweakly.

We obtain a converse to Lemma 5.

Corollary 8. *Let G be a locally compact group,* $1 < p < \infty$ *and* $T \in \mathcal{L}(L^p(G))$. *Suppose that*

$$\sum_{n=1}^{\infty}\Big\langle T[\tau_p k_n],[\tau_{p'} l_n]\Big\rangle = \sum_{n=1}^{\infty}\Big\langle T[\tau_p k'_n],[\tau_{p'} l'_n]\Big\rangle$$

for every $\Big((k_n),(l_n)\Big), \Big((k'_n),(l'_n)\Big) \in \mathcal{A}_p(G)$ *with* $\sum_{n=1}^{\infty}\overline{k}_n * \check{l}_n = \sum_{n=1}^{\infty}\overline{k'_n} * \check{l}'_n$. *Then* $T \in PM_p(G)$.

Proof. Let $u \in A_p(G)$. We set

$$F(u) = \sum_{n=1}^{\infty}\overline{\Big\langle T[\tau_p k_n],[\tau_{p'} l_n]\Big\rangle}$$

for every $\Big((k_n),(l_n)\Big) \in \mathcal{A}_p(G)$ with $u = \sum_{n=1}^{\infty}\overline{k}_n * \check{l}_n$. Clearly $F \in A_p(G)'$. Theorem 6 implies the existence of $S \in PM_p(G)$ with

$$F(\overline{\varphi} * \check{\psi}) = \overline{\Big\langle S[\tau_p\varphi],[\tau_{p'}\psi]\Big\rangle}$$

for every $\varphi \in \mathcal{L}^p(G)$ and every $\psi \in \mathcal{L}^{p'}(G)$. Then $S = T$.

Definition 4. Let G be a locally compact group and $1 < p < \infty$. For every $T \in PM_p(G)$ and $u \in A_p(G)$ we put:

$$\langle u, T\rangle_{A_p,PM_p} = \Psi_G^p(T)(u).$$

Remark. Clearly $<,>_{A_p,PM_p}$ is a sesquilinear form on $A_p(G) \times PM_p(G)$.

4.2 Applications to Abelian Groups

We show that $CV_2(G) = PM_2(G)$ for G abelian, without using the approximation theorem of Sect. 2.3.

Theorem 1. *Let G be a locally compact abelian group. Then* $CV_2(G) = PM_2(G)$ *and we have*

$$\langle u, T\rangle_{A_2,PM_2} = \Big\langle \Phi_{\hat{G}}^{-1}(u), \widehat{T}\Big\rangle$$

for every $T \in CV_2(G)$ *and for every* $u \in A_2(G)$.

Proof. To prove that $T \in PM_2(G)$ it will be enough (see Corollary 8 Sect. 4.1) to verify the equality

$$\sum_{n=1}^{\infty}\left\langle T[\check{k_n}],[\check{l_n}]\right\rangle=\left\langle\Phi_{\hat{G}}^{-1}(u),\widehat{T}\right\rangle$$

for every $\Big((k_n),(l_n)\Big)$ with $u=\sum_{n=1}^{\infty}\overline{k}_n * \check{l}_n$.

We put $r_n=\mathcal{F}(k_n)$ and $s_n=\mathcal{F}(l_n)$. By Remark (1) of Theorem 1 of Sect. 1.3 we have, for $N\in\mathbb{N}$, we have

$$\sum_{n=1}^{N}\overline{\left\langle T[\check{k_n}],[\check{l_n}]\right\rangle}=\left\langle\sum_{n=1}^{N}\tilde{r}_n\check{s}_n,\widehat{T}\right\rangle,$$

but by Lemma 1 of Sect. 3.2 we have

$$\lim\left\|\sum_{n=1}^{N}\tilde{r}_n\check{s}_n-\Phi_{\hat{G}}^{-1}(u)\right\|_1=0$$

and therefore

$$\sum_{n=1}^{\infty}\left\langle T[\check{k_n}],[\check{l_n}]\right\rangle=\left\langle\Phi_{\hat{G}}^{-1}(u),\widehat{T}\right\rangle.$$

So $T\in PM_2(G)$. But then by the definitions of Sect. 4.1

$$\left\langle u,T\right\rangle_{A_2,PM_2}=\left\langle\Phi_{\hat{G}}^{-1}(u),\widehat{T}\right\rangle,$$

which ends the proof of the theorem.

Definition 1. Let G be a locally compact abelian group. For every $\varphi\in L^{\infty}(\widehat{G})$ and every $f\in L^1(\widehat{G})$ we set

$$\Theta_{\hat{G}}(\varphi)(f)=\left\langle f,\varphi\right\rangle.$$

The map $\Theta_{\hat{G}}$ is a conjugate linear isometry of $L^{\infty}(\widehat{G})$ onto $L^1(\widehat{G})'$. Let ${}^t\Phi_{\hat{G}}$ be the transposed of the map $\Phi_{\hat{G}}$. Then

$$\Lambda_{\hat{G}}^{-1}=\Theta_{\hat{G}}^{-1}\circ{}^t\Phi_{\hat{G}}\circ\Psi_G^2.$$

Indeed, for $f\in L^1(\widehat{G})$ and $\varphi\in L^{\infty}(\widehat{G})$ we put $u=\Phi_{\hat{G}}(f)$ and $T=\Lambda_{\hat{G}}(\varphi)$ in the formula of Theorem 1, we get

$$\left\langle f,\varphi\right\rangle=\left\langle\Phi_{\hat{G}}(f),\Lambda_{\hat{G}}(\varphi)\right\rangle_{A_2,PM_2}=\Psi_G^2(\Lambda_{\hat{G}}(\varphi))(\Phi_{\hat{G}}(f))={}^t\Phi_{\hat{G}}\Big(\Psi_G^2(\Lambda_{\hat{G}}(\varphi))\Big)(f)$$

and thus

$${}^t\Phi_{\hat{G}}\Big(\Psi_G^2(\Lambda_{\hat{G}}(\varphi))\Big)=\Theta_{\hat{G}}(\varphi).$$

The following result completes Theorem 2 of Sect. 1.3.

Corollary 3. *Let G be a locally compact abelian group. Then $\Lambda_{\hat{G}}$ is a homeomorphism of $L^\infty(\widehat{G})$, with the topology $\sigma(L^\infty, L^1)$, onto $CV_2(G)$ with the ultraweak topology.*

Corollary 4. *Let G be a locally compact abelian group and $u \in \mathcal{L}^\infty(\widehat{G})$. Then there is a net (v_α) of trigonometric polynorms such that:*

1. $\|v_\alpha\|_\infty \leq \|u\|_\infty$ *for every* α,
2. $\lim\limits_\alpha v_\alpha = u$ *for the topology* $\sigma(L^\infty, L^1)$.

Corollary 5. *Let G be an infinite locally compact abelian group. Then*

$$CV_2(G) \neq \lambda_G^2(M^1(G)).$$

Proof. Suppose that $CV_2(G) = \lambda_G^2(M^1(G))$. For $f \in L^1(\widehat{G})$ consider $\varphi(f) = \widehat{f}\circ\varepsilon_G$. Then φ is a linear continuous map of $L^1(\widehat{G})$ into $C_0(G)$ with $\|\varphi(f)\|_u \leqslant \|f\|_1$. We show that ${}^t\varphi$ is a surjective map of $C_0(G)'$ onto $L^1(\widehat{G})'$.

Let $F \in L^1(\widehat{G})'$. Then $\Theta_{\hat{G}}^{-1}(F) \in L^\infty(\widehat{G})$ and $\Lambda_{\hat{G}}(\Theta_{\hat{G}}^{-1}(F)) \in CV_2(G)$. There is $\mu \in M^1(G)$ with $\Lambda_{\hat{G}}(\Theta_{\hat{G}}^{-1}(F)) = \lambda_G^2(\mu)$. Consequently $\Theta_{\hat{G}}^{-1}(F) = ((\widehat{\mu})^\vee)^\cdot$. Then $({}^t\varphi)(\overline{\mu}) = F$. The map ${}^t\varphi$ is injective: let indeed $\nu \in M^1(G)$ with $({}^t\varphi)(\nu)=0$. For every $f \in \mathcal{L}^1(\widehat{G})$ we have

$$\nu(\widehat{f}\circ\varepsilon_G) = \int_{\hat{G}} f(\chi)\widehat{\nu}(\chi)d\chi = 0$$

thus $\widehat{\nu} = 0$ and consequently $\nu = 0$.

From 3 Lemma of VI.6. (p. 488) of [38] it follows that $\varphi(L^1(\widehat{G})) = C_0(G)$. According [105] Theorem 5.4.5, p. 161 the group G is finite.

Remark. 1. One can show that for every $1 < p < \infty$ and for every infinite locally compact abelian group G one has $CV_p(G) \neq CV_2(G)$ (see Larsen [73], Theorem 4.5.2., p. 110.)
2. Using the results of this book, the following generalization of Corollary 5 can be easily obtained: if a locally compact group G admits an infinite abelian subgroup then $CV_2(G) \neq \lambda_G^2(M^1(G))$.

Theorem 6. *Let V be a Banach space and ω a linear functional of V'. Then the following statements are equivalent:*

1. *There is $v \in V$ with $\omega(F) = F(v)$ for every $F \in V'$,*
2. *For every net (F_i) of V' and $F \in V'$ with* $\lim F_i = F$ *for $\sigma(V', V)$ and $\|F_i\| \leq C$ for $C > 0$ we have* $\lim \omega(F_i) = \omega(F)$.

Proof. See 6 Theorem of Chap. V.5., p. 428 of [38].

Theorem 7. *Let G be a locally compact abelian group and $1 < p < \infty$. Then*

1. *We have $A_2(G) \subset A_p(G)$, and $\|u\|_{A_p} \le \|u\|_{A_2}$ for every $u \in A_2(G)$.*
2. *For $T \in PM_p(G)$ and $u \in A_2(G)$ we also have*

$$\langle u, T\rangle_{A_p,PM_p} = \langle u, \alpha_p(T)\rangle_{A_2,PM_2}.$$

Proof. Let $u \in A_2(G)$. There is (Sect. 3.2, Theorem 2) $k, l \in \mathcal{L}^2(G)$ with $u = \overline{k} * \check{l}$ and $\|u\|_{A_2} = N_2(k)N_2(l)$. For every $F \in A_p(G)'$ we set

$$\omega(F) = \overline{\left\langle \alpha_p\Big((\Psi_G^p)^{-1}(F)\Big)[\check{k}],\ [\check{l}]\right\rangle}.$$

Let (F_i) be a net of $A_p(G)'$, $F \in A_p(G)'$ and $K > 0$ such that $\lim F_i = F$ for the topology $\sigma(A_p', A_p)$ and with $\|F_i\|_{A_p'} \le K$ for all i. Then

$$\left\|\left|(\Psi_G^p)^{-1}(F_i)\right|\right\|_p \le K$$

for every i, and

$$\lim(\Psi_G^p)^{-1}(F_i) = (\Psi_G^p)^{-1}(F)$$

for the ultraweak operator topology on $CV_p(G)$.

In particular if $r, s \in \mathcal{M}_{00}^\infty(G)$ we have

$$\lim\left\langle(\Psi_G^p)^{-1}(F_i)[r], [s]\right\rangle = \left\langle(\Psi_G^p)^{-1}(F)[r], [s]\right\rangle.$$

From $[r] \in L^2(G) \cap L^p(G)$ we get $\alpha_p(S)[r] = S[r]$ for $S \in PM_p(G)$ and therefore

$$\lim\left\langle\alpha_p\Big((\Psi_G^p)^{-1}(F_i)\Big)[r], [s]\right\rangle = \left\langle\alpha_p\Big((\Psi_G^p)^{-1}(F)\Big)[r], [s]\right\rangle.$$

Moreover we have

$$\left\|\left|\alpha_p\Big((\Psi_G^p)^{-1}(F_i)\Big)\right|\right\|_2 \le K$$

for every $i \in I$. By the second remark to Definition 1 of Sect. 4.1 this implies that

$$\lim \alpha_p\Big((\Psi_G^p)^{-1}(F_i)\Big) = \alpha_p\Big((\Psi_G^p)^{-1}(F)\Big)$$

for the ultraweak operator topology on $PM_2(G)$. In particular

$$\lim\left\langle\alpha_p\Big((\Psi_G^p)^{-1}(F_i)\Big)[\check{k}], [\check{l}]\right\rangle = \left\langle\alpha_p\Big((\Psi_G^p)^{-1}(F)\Big)[\check{k}], [\check{l}]\right\rangle$$

and therefore $\lim \omega(F_i) = \omega(F)$. According to Theorem 6, there is a $v \in A_p(G)$ with $\omega(F) = F(v)$ for every $F \in A_p(G)'$.

For $F = \Psi_G^p(T)$ with $T \in PM_p(G)$ we hence obtain

$$\Psi_G^p(T)(v) = \omega\Big(\Psi_G^p(T)\Big) = \overline{\Big\langle \alpha_p(T)\Big[\check{k}\Big], \Big[\check{l}\Big]\Big\rangle} = \Psi_G^2\big(\alpha_p(T)\big)(\overline{k} * \check{l}),$$

and in particular for $\mu \in M^1(G)$:

$$\Psi_G^p\big(\lambda_G^p(\mu)\big)(v) = \Psi_G^2\big(\alpha_p(\lambda_G^p(\mu))\big)(\overline{k} * \check{l}) = \Psi_G^2\big(\alpha_p(\lambda_G^p(\mu))\big)(u).$$

From $\alpha_p(\lambda_G^p(\mu)) = \lambda_G^2(\mu)$ and $\Psi_G^p\big(\lambda_G^p(\mu)\big) = \tilde{\mu}$ we deduce $\tilde{\mu}(v) = \tilde{\mu}(u)$ and therefore $u = v$.

This implies $u \in A_p(G)$ and

$$\big\langle u, T\big\rangle_{A_p, PM_p} = \big\langle u, \alpha_p(T)\big\rangle_{A_2, PM_2}.$$

Consequently

$$\|u\|_{A_p} \leqslant \|u\|_{A_2}.$$

Remarks. 1. For $u \in A_2(G)$ and $T \in PM_p(G)$ we have

$$\big\langle u, T\big\rangle_{A_p, PM_p} = \Theta_{\hat{G}}(\widehat{T})\Big(\Phi_{\hat{G}}^{-1}(u)\Big).$$

2. Clearly the map α_p is the adjoint of the inclusion of $A_2(G)$ in $A_p(G)$.
3. We will generalize this theorem to the class of amenable groups in Sect. 8.3.

4.3 Holomorphic Functions Operating on $A_p(G)$

In analogy with the case of the Fourier algebra of a locally compact abelian group ([105] Chap. 6), we investigate whether in the case of an arbitrary locally compact group G the composed $F \circ u$ of $u \in A_p(G)$ and the holomorphic function $F \in \mathbb{C}^U$ still belongs to $A_p(G)$.

Observe at first that for G a locally compact group, $1 < p < \infty$, $a \in G$ and $u \in A_p(G)$ we have ${}_a u, u_a \in A_p(G)$ and $\|{}_a u\|_{A_p} = \|u_a\|_{A_p} = \|u\|_{A_p}$.

Lemma 1. *Let G be a locally compact group,* $1 < p < \infty$ *and* $u \in A_p(G)$. *Then* $x \mapsto_x u$ $A_p(G)$ *and* $x \mapsto u_x$ *are continuous maps of G into* $A_p(G)$.

Proof. We check only the first statement.

Let $\varepsilon > 0$ and $x_0 \in G$, and let $u = \sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n$ with $\Big((k_n), (l_n)\Big) \in \mathcal{A}_p(G)$.

Choose $M \in \mathbb{N}$ such that

$$\sum_{n=M+1}^{\infty} N_p(k_n) N_{p'}(l_n) < \frac{\varepsilon}{4}.$$

There is an open neighborhood U of x_0 with

$$N_p({}_xk_n - {}_{x_0}k_n) < \frac{\varepsilon}{2M\big(1 + N_{p'}(l_n)\big)}$$

for $x \in U$ and $1 \le n \le M$.

For every $x \in U$ we then have

$$\begin{aligned}\|{}_xu - {}_{x_0}u\|_{A_p} &\le \sum_{n=1}^{\infty} N_p({}_xk_n - {}_{x_0}k_n)N_{p'}(l_n)\\ &< \frac{\varepsilon}{2} + 2\sum_{n=M+1}^{\infty} N_p(k_n)N_{p'}(l_n)\end{aligned}$$

and consequently $\|{}_xu - {}_{x_0}u\|_{A_p} < \varepsilon$.

Theorem 2. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$, $a \in G$ and $\varepsilon > 0$. Suppose that $u(a) = 0$. Then there is $v \in A_p(G) \cap C_{00}(G)$, vanishing on a neighborhood of a, and such that $\|u - v\|_{A_p} < \varepsilon$.*

Proof. According to Corollary 7 of Sect. 3.1 there is $w \in A_p(G) \cap C_{00}(G)$ with

$$\|u - w\|_{A_p} < \frac{\varepsilon}{6}.$$

By Lemma 1 above e admits an open relatively compact neighborhood V such that for every $x \in V$ we have

$$\|w - w_x\|_{A_p} < \frac{\varepsilon}{6}.$$

For $y \in aV$ we have therefore

$$|w(y)| = |u(a) - w(aa^{-1}y)| = |u(a) - w_{a^{-1}y}(a)| \le \|u - w_{a^{-1}y}\|_{A_p} < 2\varepsilon/6.$$

Choose a compact set $K \subset V$ such that $m(K) > m(V)/2$, and put

$$k = \frac{1_K}{m(K)},\ l = w1_{aV}, \quad v = w * \check{k} - l * \check{k}.$$

Then $v \in A_p(G) \cap C_{00}(G)$ and

$$\begin{aligned}\|u - v\|_{A_p} &< \|u - w\|_{A_p} + \|w - w * \check{k}\|_{A_p} + \|l * \check{k}\|_{A_p}\\ &< \frac{\varepsilon}{6} + \|w - w * \check{k}\|_{A_p} + \|l * \check{k}\|_{A_p}.\end{aligned}$$

To estimate $\|w - w * \check{k}\|_{A_p}$, observe that for every $T \in PM_p(G)$

$$\langle w - w * \check{k}, T\rangle_{A_p,PM_p} = \int_G k(y)\langle w - w_y, T\rangle_{A_p,PM_p} dy,$$

Theorem 6 of Sect. 4.1 implies then

$$\|w - w * \check{k}\|_{A_p} \leq \frac{\varepsilon}{6}.$$

From

$$N_{p'}(k) = \frac{1}{m(K)^{1/p}} < \frac{2}{m(V)^{1/p}}$$

and

$$N_p(l) = \left(\int_{aV} |w(x)|^p dx\right)^{1/p} \leq \frac{\varepsilon}{3} m(V)^{1/p}$$

we obtain

$$\|l * \check{k}\|_{A_p} \leq N_p(l) N_{p'}(k) \leq \frac{2\varepsilon}{3}$$

and finally

$$\|u - v\|_{A_p} < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{2\varepsilon}{3} = \varepsilon.$$

Observe finally that v vanishes on aW, where W is an open neighborhood of e such that $WK \subset V$.

Remark. For $p = 2$ this result is due to Eymard ([41] (4.11) Corollaire 2, p. 229). For p arbitrary the result is due to Herz ([61], Theorem B, p. 91).

Proposition 3. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$, $a \in G$, U a neighborhood of a, $\alpha > 1$ and $\varepsilon > 0$. Then there is $v \in A_p(G) \cap C_{00}(G)$ and V neighborhood of a such that:*

1. $0 \leq v(x) \leq 1$ *for every* $x \in G$,
2. $v(x) = 1$ *on* V,
3. $\|v\|_{A_p} < \alpha$,
4. $\operatorname{supp} v \subset U$,
5. $\|uv - u(a)v\|_{A_p} < \varepsilon$.

Proof. Without loss of generality we can assume that $a = e$.

First suppose that $u(e) = 0$. By Theorem 2, there is $v \in A_p(G) \cap C_{00}(G)$ and an open set U_1 neighborhood of e with $v(y) = 0$ on U_1 and

$$\|u - v\|_{A_p} < \frac{\varepsilon}{\alpha}.$$

There are W, K compact neighborhoods of e with $K = K^{-1}$ and $WK^2 \subset U_1 \cap U$. Let W' be a compact neighborhood of e with $W'K \subset U_2$ where U_2 is an open subset of G such that $K \subset U_2$ and such that $m(U_2) < \alpha^p m(K)$. We put finally

$$w = \frac{1_{VK} * 1_K}{m(K)}$$

where $V = W \cap W'$. We have $0 \leq w(x) \leq 1$, $w(x) = 1$ on V, $vw = 0$, $\operatorname{supp} w \subset U$,

$$\|w\|_{A_p} < \alpha \quad \text{and} \quad \|uw\|_{A_p} = \|(u-v)w\|_{A_p} < \varepsilon.$$

Suppose that $u(e) \neq 0$.

There is $v \in A_p(G) \cap C_{00}(G)$ with

$$\|u - v\|_{A_p} < \frac{\varepsilon_1}{3\alpha} \quad \text{where} \quad 0 < \varepsilon_1 < \min\Big\{\varepsilon, 3\alpha|u(e)|\Big\}.$$

Then $\operatorname{supp} v$ is a compact neighborhood of e. Let $v_1 \in A_p(G) \cap C_{00}(G)$ with $0 \leq v_1(x) \leq 1$ for every $x \in G$ and $v_1(x) = 1$ on $\operatorname{supp} v$. We have

$$vv_1 - v(e)v_1 \in A_p(G) \quad \text{and} \quad (vv_1 - v(e)v_1)(e) = 0.$$

By the first part of the proof there is $w \in A_p(G) \cap C_{00}(G)$ and V a neighborhood of e with $0 \leq w(x) \leq 1$ for every $x \in G$, $w(x) = 1$ on V, $\|w\|_{A_p} < \alpha$, $\operatorname{supp} w \subset \operatorname{supp} v \cap U$ and

$$\|(vv_1 - v(e)v_1)w\|_{A_p} < \frac{\varepsilon}{3}.$$

We get therefore

$$\|uw - u(e)w\|_{A_p} < \|wv - v(e)w\|_{A_p} + \frac{2\varepsilon}{3}$$

but taking into account $w = v_1 w$, we conclude that

$$\|uw - u(e)w\|_{A_p} < \varepsilon.$$

Remark. For G abelian and $p = 2$ see [105], Proposition 5.2.6, p. 158.

Lemma 4. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$, $a \in G$ and $F \in \mathbb{C}^U$ holomorphic on U, open neighborhood of $u(a)$ in $\mathbb{C}$. There is $w \in A_p(G)$ with $F(u(x)) = w(x)$ on a neighborhood of a.*

Proof. There is $\varepsilon > 0$ and $(c_n)_{n=1}^{\infty}$ a sequence of $\mathbb{C}$ such that:

1. $z \in U$ for every $z \in \mathbb{C}$ with $|z - u(a)| < \varepsilon$,
2. For every $z \in \mathbb{C}$ with $|z - u(a)| < \varepsilon$ we have

$$F(z) = F(u(a)) + \sum_{n=1}^{\infty} c_n (z - u(a))^n.$$

By Proposition 3 there is $v \in A_p(G) \cap C_{00}(G)$ and V neighborhood of a with $v = 1$ on V and $\|uv - u(e)v\|_{A_p} < \varepsilon$. This implies for $x \in V$

$$F(u(x)) = F(u(a)) + \sum_{n=1}^{\infty} c_n (u(x) - u(a))^n$$

and the existence of $w \in A_p(G)$ with

$$\lim_{n\to\infty} \left\| w - \left\{ F(u(a))v + \sum_{k=1}^{n} c_k (uv - u(a)v)^k \right\} \right\|_{A_p} = 0.$$

Consequently for every $x \in V$ we have $w(x) = F(u(x))$.

Theorem 5. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$, K a compact subset of G and $F \in \mathbb{C}^U$ a holomorphic function on an open U neighborhood of $u(K)$ in $\mathbb{C}$. There is $v \in A_p(G)$ with $F(u(x)) = v(x)$ for every $x \in K$.*

Proof. By Lemma 4 for every $x \in K$ there is $V_{(x)}$ neighborhood of x and $v_{(x)} \in A_p(G)$ such that for every $x' \in V_{(x)}$ we have $u(x') \in U$ and $F(u(x')) = v_{(x)}(x')$. For every $x \in K$ there is $W_{(x)}$, compact neighborhood of x with $W_{(x)} \subset V_{(x)}$ and $w_{(x)} \in A_p(G) \cap C_{00}(G)$ with $0 \le w_{(x)}(y) \le 1$ for $y \in G$, $w_{(x)}$ equal to 1 on $W_{(x)}$ and with $\operatorname{supp} w_{(x)} \subset V_{(x)}$. There is also $x_1, \ldots, x_N \in K$ such that $K \subset \bigcup_{j=1}^{N} W_{(x_j)}$. Then we put:

$$v_j = v_{(x_j)}, w_j = w_{(x_j)} \quad \text{for} \quad 1 \le j \le N,$$

$$h_1 = w_1, h_j = w_j \prod_{r=1}^{j-1} (1_G - w_r) \quad \text{for every } 2 \le j \le N,$$

$$w = \sum_{j=1}^{N} h_j v_j.$$

Then $h_j \in A_p(G)$ and consequently $w \in A_p(G)$. For every $x \in K$ we have

$$\sum_{j=1}^{N} h_j(x) = 1 \quad \text{and} \quad h_j(x)v_j(x) = h_j(x)F(u(x)) \quad \text{for every} \quad 1 \le j \le N.$$

We finally obtain $w(x) = F(u(x))$ for every $x \in K$.

Remarks. 1. For $G = \mathbb{R}$, $p = 2$ and compact intervals see Carleman [14], Chap. IV, p. 67.

2. For G a locally compact abelian group and $p = 2$ see Reiter and Stegeman [105], Chap. 6, Theorem 6.1.1, p. 169.

The following corollary will be used in Sect. 6.2.

Corollary 6. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$ and K a compact subset of G. Suppose that $u(x) \neq 0$ for every $x \in K$. Then there is $v \in A_p(G)$ such that $v(x) = \dfrac{1}{u(x)}$ for every $x \in K$.*

Remarks. 1. It follows that $A_p(G)$ is a normed standard algebra in Reiter's sense ([105], Definition 2.2.5, p. 29).

2. As a consequence of 1) we obtain ([105] Proposition 2.1.14 p. 28) the following result: let I be an arbitrary ideal in $A_p(G)$, then I contains every $u \in A_p(G) \cap C_{00}(G)$ such that $\operatorname{supp} u \cap cospI = \emptyset$.

Corollary 7. *Let G be a compact group, $1 < p < \infty$, $u \in A_p(G)$ and $F \in \mathbb{C}^U$ a holomorphic function on an open neighborhood U of $u(G)$ in $\mathbb{C}$. Then $F \circ u \in A_p(G)$.*

Remark. For $p = 2$ see Hewitt and Ross [67] (39.31) Theorem.

Theorem 8. *Let G be a locally compact non-compact group, $1 < p < \infty$, $u \in A_p(G)$ and $F \in \mathbb{C}^U$, a holomorphic function on an open neighborhood U of $u(G)$ in $\mathbb{C}$. Suppose that $0 \in U$ and that $F(0) = 0$. Then $F \circ u$ belongs to $A_p(G)$.*

Proof. There is $\varepsilon > 0$ and $(c_n)_{n=1}^{\infty}$ a sequence of $\mathbb{C}$ such that:

1. $z \in U$ for every $z \in \mathbb{C}$ with $|z| < \varepsilon$,
2. for every $z \in \mathbb{C}$ with $|z| < \varepsilon$ we have

$$F(z) = \sum_{n=1}^{\infty} c_n z^n.$$

There is $v \in A_p(G) \cap C_{00}(G; \mathbb{C})$ with $\|u - v\|_{A_p(G)} < \varepsilon$. There is $b_0 \in A_p(G)$ such that

$$\lim_{n \to \infty} \Big\| b_0 - \sum_{k=1}^{n} c_k (u - v)^k \Big\|_{A_p} = 0.$$

This implies $b_0 = F \circ (u - v)$. Let $\varphi \in A_p(G) \cap C_{00}(G)$ with $\varphi = 1$ on $\operatorname{supp} v$. By Theorem 5 there is $b_1 \in A_p(G)$ with $b_1(x) = F(u(x))$ for every $x \in \operatorname{supp} \varphi$. Then $F \circ u = b_0 - \varphi b_0 + \varphi b_1$.

Lemma 9. *Let G be a locally compact group, $1 < p < \infty$, $a \in G$, $u \in A_p(G)$, U an open neighborhood of $u(a)$ in $\mathbb{C}$ and (f_n) a sequence of holomorphic functions on U. Suppose that (f_n) converges uniformly to 0 on every compact subset of U. Then there is an open neighborhood V of a and a sequence (w_n) of $A_p(G)$ such that:*

1. *$w_n(x) \in U$ for every $x \in V$ and for every $n \in \mathbb{N}$,*
2. *$f_n(u(x)) = w_n(x)$ for every $x \in V$ and for every $n \in \mathbb{N}$,*
3. $\lim \|w_n\|_{A_p} = 0$.

Proof. Let $0 < \rho < \infty$ such that for every $z \in \mathbb{C}$ with

$$|z - u(a)| \leq \rho$$

we have $z \in U$. Let also $0 < r < \rho$. By Proposition 3 there is $v \in A_p(G) \cap C_{00}(G; \mathbb{R})$ and V neighborhood of a with $v(x) = 1$ on V and

$$\|uv - u(a)v\|_{A_p} < r.$$

For every $x \in V$ and every $k \in \mathbb{N}$ we have therefore

$$f_k(u(x)) = \sum_{n=0}^{\infty} \frac{f_k^{(n)}(u(a))}{n!}(u(x) - u(a))^n.$$

For every $k \in \mathbb{N}$ there is $w_k \in A_p(G)$ with

$$\lim_{n \to \infty} \left\| w_k - \left\{ f_k(u(a))v + \sum_{j=1}^{n} \frac{f_k^{(j)}(u(a))}{j!}(uv - u(a)v)^j \right\} \right\|_{A_p} = 0.$$

Then for every $x \in V$ we have $w_k(x) = f_k(u(x))$. But

$$\|w_k\|_{A_p} \leq \max \left\{ |f_k(z)| \middle| z \in \mathbb{C}, |z - u(a)| = \rho \right\} \left(\|v\|_{A_p} + \sum_{n=1}^{\infty} \left(\frac{r}{\rho}\right)^n \right)$$

and therefore

$$\|w_k\|_{A_p} \leq \max \left\{ |f_k(z)| \middle| z \in \mathbb{C}, |z - u(a)| = \rho \right\} \left(\|v\|_{A_p} + \frac{r}{\rho - r} \right).$$

Theorem 10. *Let G be a locally compact group, $1 < p < \infty$, K a compact subset of G, $u \in A_p(G)$, U an open neighborhood of $u(K)$ in $\mathbb{C}$ and (f_n) a sequence of holomorphic functions on U. Suppose that (f_n) converges uniformly to 0 on every compact subset of U. Then there is a sequence (a_n) of $A_p(G)$ with $f_n(u(x)) = a_n(x)$ for every $x \in K$ and with* $\lim \|a_n\|_{A_p} = 0$.

Proof. By Lemma 9 for every $x \in K$ there is $V_{(x)}$, open neighborhood of x, and $\left(v_n^{(x)}\right)$ a sequence of $A_p(G)$ such that:

1. $v_n^{(x)}(x') \in U$ for every $x' \in V_{(x)}$ and for every $n \in \mathbb{N}$,
2. $f_n(u(x')) = v_n^{(x)}(x')$ for every $x' \in V_{(x)}$ and for every $n \in \mathbb{N}$,
3. $\lim \|v_n^{(x)}\|_{A_p} = 0$.

For $x \in K$ let $W_{(x)}$ be a compact neighborhood of x with $W_{(x)} \subset V_{(x)}$ and $w_{(x)} \in A_p(G) \cap C_{00}(G; \mathbb{R})$ such that $0 \leq w_{(x)}(y) \leq 1$ for every $y \in G$, $w_{(x)} = 1$ on $W_{(x)}$

and such that $\operatorname{supp} w_{(x)} \subset V_{(x)}$. There is finally $N \in \mathbb{N}$ and $x_1, \ldots, x_N \in K$ such that $K \subset W_{(x_1)} \cup \ldots \cup W_{(x_N)}$. We put: $w_j = w_{x_j}$ and $v_n^{(j)} = v_n^{(x_j)}$ for $1 \leq j \leq N$ and $n \in \mathbb{N}$. We also put $h_1 = w_1$ and $h_j = w_j \prod_{r=1}^{j-1}(1_G - w_j)$ for $2 \leq j \leq N$. For every $x \in K$ we have

$$\sum_{j=1}^{N} h_j(x) = 1 \quad \text{and} \quad h_j(x)v_n^{(j)}(x) = h_j(x) f_n(u(x)) \quad \text{for every } 1 \leq j \leq N$$

and for every $n \in \mathbb{N}$. Then for every $x \in K$ $a_n(x) = f_n(u(x))$ with $a_n = \sum_{j=1}^{N} h_j v_n^{(j)}$ for every $n \in \mathbb{N}$. Clearly $\lim \|a_n\|_{A_p} = 0$.

Corollary 11. *Let G be a compact group, $1 < p < \infty$, $u \in A_p(G)$, U an open neighborhood of $u(G)$ in $\mathbb{C}$ and (f_n) a sequence of holomorphic functions on U. Suppose that (f_n) converges uniformly to 0 on every compact subset of U. According to Corollary 7 $f_n \circ u \in A_p(G)$ for every $n \in \mathbb{N}$. We have* $\lim \|f_n \circ u\|_{A_p} = 0$.

Remark. For $p = 2$ and $G = \mathbb{T}$ see Zygmund [121], Vol. I, Chap. VI, p. 246 (5.7) Theorem.

Theorem 12. *Let G be a locally compact non-compact group, $1 < p < \infty$, $u \in A_p(G)$ and (f_n) a sequence of holomorphic functions on an open neighborhood U of $u(G)$ in $\mathbb{C}$. Suppose that $0 \in U$, $f_n(0) = 0$ for every $n \in \mathbb{N}$ and that the sequence (f_n) converges uniformly to 0 on every compact subset of U. According to Theorem 8 $f_n \circ u$ belongs to $A_p(G)$ for every $n \in \mathbb{N}$. We have* $\lim \|f_n \circ u\|_{A_p} = 0$.

Proof. Let $0 < \rho < \infty$ such that for every $z \in \mathbb{C}$ with $|z| \leq \rho$ we have $z \in U$. Choose $0 < r < \rho$. There is $v \in A_p(G) \cap C_{00}(G)$ with

$$\|u - v\|_{A_p} < r.$$

For every $k \in \mathbb{N}$ there is $b_k \in A_p(G)$ with

$$\lim_{n\to\infty} \left\| b_k - \sum_{j=1}^{n} \frac{f_k^{(j)}(0)}{j!}(u - v)^j \right\|_{A_p} = 0.$$

For every $x \in G$ we have $b_k(x) = f_k(u(x) - v(x))$. But

$$\|b_k\|_{A_p} \leq \max\left\{|f_k(z)| \,\Big|\, |z| = \rho\right\}\left(\frac{r}{\rho - r}\right)$$

and therefore $\lim \|b_k\|_{A_p} = 0$. As in the proof of Theorem 8, choose $\varphi \in A_p(G) \cap C_{00}(G; \mathbb{R})$ with $\varphi(x) = 1$ for every $x \in \operatorname{supp} v$. By Theorem 10 there is a sequence (c_k) of $A_p(G)$ with $c_k(x) = f_k(u(x))$ for every $x \in \operatorname{supp}\varphi$ and such that $\lim \|c_k\|_{A_p} = 0$. Let finally $d_k = b_k - \varphi b_k + \varphi c_k$ for every $k \in \mathbb{N}$. We have $d_k \in A_p(G)$, $d_k(x) = f_k(u(x))$ for every $x \in G$ and $\lim \|d_k\|_{A_p} = 0$.

Corollary 13. *Let G be a locally compact group, $1 < p < \infty$ and $u \in A_p(G)$. Then:*

1. *The sequence $\left(\|u^n\|_{A_p}^{1/n}\right)$ converges,*
2. $\lim_{n\to\infty} \|u^n\|_{A_p}^{1/n} = \max\left\{|u(x)|\,|\,x \in G\right\}$.

Proof. Proposition 1 of [8] (Chap. I, Sect. 2, no. 3, p. 15) implies 1. For every $n \in \mathbb{N}$ we have

$$\max\left\{|u^n(x)|\,|\,x \in G\right\} = \max\left\{|u(x)|\,|\,x \in G\right\}^n$$

and therefore

$$\lim_{n\to\infty} \|u^n\|_{A_p}^{1/n} \geq \max\left\{|u(x)|\,|\,x \in G\right\}.$$

Now let $R \in \mathbb{R}$ with

$$R > \max\left\{|u(x)|\,|\,x \in G\right\}.$$

For $n \in \mathbb{N}$ and $z \in \mathbb{C}$ with $|z| < R$ define $f_n(z) = \dfrac{z^n}{R^n}$. We have

$$\lim \|f_n \circ u\|_{A_p} = 0$$

and therefore

$$\lim_{n\to\infty} \|u^n\|_{A_p}^{1/n} \leq R.$$

Remarks. 1. This proof is directly inspired from Zygmund ([121], Vol. I, Chap. VI (5.8) Theorem, p. 246).

2. For $p = 2$ and G abelian see Hewitt and Ross ([67], Chap. X, Sect. 39, p. 519).

Chapter 5
$CV_p(G)$ as a Module on $A_p(G)$

By the map $(f, g) \mapsto f * g$, $L^\infty(G)$ is a left module on $L^1(G)$. Similarly for $u \in A_p(G)$ and $T \in CV_p(G)$ we define a convolution operator uT. With the map $(u, T) \mapsto uT$ $CV_p(G)$ is a left module on $A_p(G)$. For $f \in L^1(G)$ and $g \in L^\infty(G)$ the function $f * g$ is more regular than the function g. Similarly the convolution operator uT is a smoothing of the convolution operator T. In particular $uT \in PM_p(G)$. For G amenable we obtain a generalization of the approximation theorem for $CV_p(G)$.

5.1 The Convolution Operator $(\overline{k} * \check{l})T$

Lemma 1. *Let G be a locally compact group, $1 < p < \infty$, $\varphi \in \mathcal{M}_{00}^\infty(G)$ and $k \in \mathcal{L}^p(G)$. Then the map $G \to \mathcal{L}^p(G)$, $t \mapsto {}_{t^{-1}}(\check{k})\varphi$ is continuous, and we have*

$$\left(\int_G N_p\big({}_{t^{-1}}(\check{k})\varphi\big)^p dt\right)^{1/p} = N_p(k)N_p(\varphi).$$

Lemma 2. *Let G be a locally compact group, $1 < p < \infty$, $k \in \mathcal{L}^p(G)$, $l \in \mathcal{L}^{p'}(G)$, $\varphi, \psi \in \mathcal{M}_{00}^\infty(G)$ and $T \in CV_p(G)$. Then the function*

$$F : G \to \mathbb{C}, t \mapsto \Big\langle T[{}_{t^{-1}}(\check{k})\varphi], [{}_{t^{-1}}(\check{l})\psi]\Big\rangle$$

belongs to $\mathcal{L}_{\mathbb{C}}^1(G) \cap C(G)$, and $N_1(F) \leq |||T|||_p\, N_p(k)N_{p'}(l)N_p(\varphi)N_{p'}(\psi)$.

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Proof. By Lemma 1 F is clearly continuous, and

$$\int_G^* |F(t)|dt \leq \|T\|_p \int_G^* N_p\Big({}_{t^{-1}}(\check{k})\varphi\Big) N_{p'}\Big({}_{t^{-1}}(\check{l})\psi\Big) dt$$

$$\leq \|T\|_p \left(\int_G N_p\Big({}_{t^{-1}}(\check{k})\varphi\Big)^p dt\right)^{1/p} \left(\int_G N_{p'}\Big({}_{t^{-1}}(\check{l})\psi\Big)^{p'} dt\right)^{1/p'}$$

$$= \|T\|_p N_p(k) N_{p'}(l) N_p(\varphi) N_{p'}(\psi).$$

Proposition 3. *Let G be a locally compact group, $1 < p < \infty$, $k \in \mathcal{L}^p(G)$, $l \in \mathcal{L}^{p'}(G)$ and $T \in CV_p(G)$. There is an unique bounded operator U of $L^p(G)$ such that*

$$\Big\langle U([\varphi]), [\psi]\Big\rangle = \int_G \Big\langle T[{}_{t^{-1}}(\check{k})\varphi], [{}_{t^{-1}}(\check{l})\psi]\Big\rangle dt$$

for every $\varphi, \psi \in \mathcal{M}_{00}^\infty(G)$. The operator U is a p-convolution operator and

$$\|U\|_p \leqslant \|T\|_p N_p(k) N_{p'}(l).$$

Proof. We verify only that U is a p-convolution operator. Let $\varphi, \psi \in \mathcal{M}_{00}^\infty(G)$ and $a \in G$. For $t \in G$ we have

$$\Big\langle T\big[{}_{t^{-1}}(\check{k})\,{}_a\varphi\big], \big[{}_{t^{-1}}(\check{l})\psi\big]\Big\rangle = \Big\langle {}_{a^{-1}}\Big(T\big[{}_{t^{-1}}(\check{k})\,{}_a\varphi\big]\Big), {}_{a^{-1}}\Big(\big[{}_{t^{-1}}(\check{l})\psi\big]\Big)\Big\rangle$$

$$= \Big\langle T\big[{}_{a^{-1}}({}_{t^{-1}}(\check{k}))\varphi\big], \big[{}_{a^{-1}}({}_{t^{-1}}(\check{l}))\,{}_{a^{-1}}\psi\big]\Big\rangle = \Big\langle T\big[{}_{(at)^{-1}}(\check{k})\varphi\big], \big[{}_{(at)^{-1}}(\check{l})\,{}_{a^{-1}}\psi\big]\Big\rangle$$

and therefore

$$\Big\langle U\big({}_a[\varphi]\big), [\psi]\Big\rangle = \int_G \Big\langle T\big[{}_{(at)^{-1}}(\check{k})\varphi\big], \big[{}_{(at)^{-1}}(\check{l})\,{}_{a^{-1}}\psi\big]\Big\rangle dt$$

$$= \int_G \Big\langle T\big[{}_{t^{-1}}(\check{k})\varphi\big], \big[{}_{t^{-1}}(\check{l})\,{}_{a^{-1}}\psi\big]\Big\rangle dt$$

$$= \Big\langle U[\varphi], {}_{a^{-1}}[\psi]\Big\rangle = \Big\langle {}_a\big(U([\varphi]), [\psi]\Big\rangle.$$

Definition 1. The convolution operator U of Proposition 3 is denoted $(\overline{k} * \check{l})T$.

Proposition 4. *Let G be a locally compact group and $1 < p < \infty$. Then:*

1. $\Big(\overline{(k + k')} * \check{l}\Big)T = (\overline{k} * \check{l})T + (\overline{k'} * \check{l})T,$
2. $\Big(\overline{k} * (l + l')^{\vee}\Big)T = (\overline{k} * \check{l})T + (\overline{k} * \check{l'})T,$

3. $\left(\overline{(\alpha k)} * \check{l}\right)T = \alpha\left((\overline{k} * \check{l})T\right) = \left((\overline{k} * (\overline{\alpha} l)\check{\ }\right)T = (\overline{k} * \check{l})(\alpha T)$,
4. $(\overline{k} * \check{l})(S + T) = (\overline{k} * \check{l})S + (\overline{k} * \check{l})T$ *for* $k, k' \in \mathcal{L}^p(G)$, $l, l' \in \mathcal{L}^{p'}(G)$, $S, T \in CV_p(G)$ *and* $\alpha \in \mathbb{C}$.

For $T = \lambda_G^p(\mu)$ the convolution operator $(\overline{k} * \check{l})T$ has a very concrete description.

Proposition 5. *Let* G *be a locally compact group,* $1 < p < \infty$, $\mu \in M^1(G)$ $k \in \mathcal{L}^p(G)$ *and* $l \in \mathcal{L}^{p'}(G)$. *We have* $(\overline{k} * \check{l})\lambda_G^p(\mu) = \lambda_G^p\left((\overline{k} * \check{l})\tilde{\ }\mu\right)$.

Proof. It suffices to show that for $k, l, \varphi, \psi \in C_{00}(G)$

$$\left\langle\left((\overline{k} * \check{l})\lambda_G^p(\mu)\right)[\varphi], [\psi]\right\rangle = \left\langle\lambda_G^p\left((\overline{k} * \check{l})\tilde{\ }\mu\right)[\varphi], [\psi]\right\rangle.$$

We have

$$\left\langle\left((\overline{k} * \check{l})\lambda_G^p(\mu)\right)[\varphi], [\psi]\right\rangle = \int_G \left\langle\lambda_G^p(\mu)\left[{}_{t^{-1}}(\check{k})\varphi\right], \left[{}_{t^{-1}}(\check{l})\psi\right]\right\rangle dt.$$

For $t \in G$, Proposition 2 of Sect. 4.1, implies

$$\left\langle\lambda_G^p(\mu)\left[{}_{t^{-1}}(\check{k})\varphi\right], \left[{}_{t^{-1}}(\check{l})\psi\right]\right\rangle = \overline{\tilde{\mu}\left(\overline{\tau_p r(t)} * (\tau_{p'} s(t))\check{\ }\right)}$$

where $r(t) = {}_{t^{-1}}(\check{k})\varphi$ and $s(t) = {}_{t^{-1}}(\check{l})\psi$. Consequently

$$\left\langle\left((\overline{k} * \check{l})\lambda_G^p(\mu)\right)[\varphi], [\psi]\right\rangle = \int_G \mu\left(\overline{\tau_{p'} s(t)} * (\tau_p r(t))\check{\ }\right)dt.$$

But

$$\int_G \mu\left(\overline{\tau_{p'} s(t)} * (\tau_p r(t))\check{\ }\right)dt = \int_G \left(\int_G \overline{\tau_{p'} s(t)} * (\tau_p r(t))\check{\ }(x)dt\right)d\mu(x)$$

with (Lemma 3 of Sect. 3.3)

$$\int_G \overline{\tau_{p'} s(t)} * (\tau_p r(t))\check{\ }(x)dt = \overline{l} * \check{k}(x)\langle\lambda_G^p(\delta_x)[\varphi], [\psi]\rangle$$

and therefore

$$\left\langle\left((\overline{k} * \check{l})\lambda_G^p(\mu)\right)[\varphi], [\psi]\right\rangle = \left\langle\lambda_G^p\left((\overline{k} * \check{l})\tilde{\ }\mu\right)[\varphi], [\psi]\right\rangle.$$

5.2 The Convolution Operator uT

Lemma 1. *Let G be a locally compact group, $1 < p < \infty$, $T \in CV_p(G)$, $k \in \mathcal{L}^p(G)$, $l \in \mathcal{L}^{p'}(G)$ and $\alpha, \varphi, \psi \in \mathcal{M}_{00}^{\infty}(G)$. Then*

$$\left\langle \left((\overline{k} * \check{l})(T\lambda_G^p(\overline{\alpha^*})) \right)[\varphi], [\psi] \right\rangle = \left\langle T\left[\alpha \Delta_G^{1/p'}\right], [\overline{k} * \check{l}\, r] \right\rangle$$

where

$$r(x) = \Delta_G(x)^{-1/p'} \left\langle \lambda_G^{p'}(\delta_x)[\psi], [\varphi] \right\rangle.$$

Proof. We have

$$\left\langle \left((\overline{k} * \check{l})(T\lambda_G^p(\overline{\alpha^*})) \right)[\varphi], [\psi] \right\rangle = \int_G \left\langle \left[{}_{t^{-1}}(\check{k})\varphi\right] * T\left[\Delta_G^{1/p'}\alpha\right], \left[{}_{t^{-1}}(\check{l})\psi\right] \right\rangle dt$$

$$= \int_G \left\langle T\left[\Delta_G^{1/p'}\alpha\right], \left[{}_{t^{-1}}(\check{k})\varphi\right]^* * \left[{}_{t^{-1}}(\check{l})\psi\right] \right\rangle dt$$

and

$$\left(\left({}_{t^{-1}}(\check{k})\varphi\right)^* * {}_{t^{-1}}(\check{l})\psi \right)(x) = \Delta_G(x)^{-1/p'} \left\langle \lambda_G^{p'}(\delta_x)\left[{}_{t^{-1}}(\check{l})\psi\right], \left[{}_{t^{-1}}(\check{k})\varphi\right] \right\rangle.$$

To finish the proof it suffices to verify that

$$\int_G \left(\left({}_{t^{-1}}(\check{k})\varphi\right)^* * {}_{t^{-1}}(\check{l})\psi \right)(x)\, dt = \left(\overline{k} * \check{l}\right)(x) r(x).$$

Lemma 2. *Let G be a locally compact group, $1 < p < \infty$, $k \in \mathcal{L}^p(G)$, $l \in \mathcal{L}^{p'}(G)$, $T \in CV_p(G)$, $\varphi \in C_{00}(G)$, and $\psi \in \mathcal{M}_{00}^{\infty}(G)$. Then for every $\alpha \in C_{00}(G)$, with $\alpha \geqslant 0$ and $\int_G \alpha(y)dy = 1$, we have*

$$\left| \left\langle \left((\overline{k} * \check{l})(T\lambda_G^p(\alpha^*)) \right)[\varphi], [\psi] \right\rangle - \left\langle \left((\overline{k} * \check{l})T \right)[\varphi], [\psi] \right\rangle \right| \leqslant \|\!|T|\!\|_p\, N_{p'}(\psi) N_{p'}(l)$$

$$\times \left\{ N_p(\varphi) \sup_{y \in \operatorname{supp}\alpha} N_p(k - {}_{y^{-1}}k) + N_p(k) \sup_{y \in \operatorname{supp}\alpha} N_p\left(\varphi - \varphi_{y^{-1}} \Delta_G(y^{-1})^{1/p}\right) \right\}.$$

Proof. By Lemma 2 of Sect. 5.1 the functions $t \mapsto \left\langle T\left[{}_{t^{-1}}(\check{k})\varphi\right], \left[{}_{t^{-1}}(\check{l})\psi\right] \right\rangle$ and $t \mapsto \left\langle \left(T\lambda_G^p(\overline{\alpha^*})\right)\left[{}_{t^{-1}}(\check{k})\varphi\right], \left[{}_{t^{-1}}(\check{l})\psi\right] \right\rangle$ are in $C(G) \cap \mathcal{L}^1(G)$. Let

$$A = \left\langle \left((\overline{k} * \check{l})(T\lambda_G^p(\overline{\alpha^*})) \right)[\varphi], [\psi] \right\rangle - \left\langle \left((\overline{k} * \check{l})T \right)[\varphi], [\psi] \right\rangle$$

then

$$A = \int_G \left\langle T\left(\left[{}_{t^{-1}}(\check{k})\varphi \right] * \left[\Delta_G^{1/p'} \alpha \right] - \left[{}_{t^{-1}}(\check{k})\varphi \right] \right), \left[{}_{t^{-1}}(\check{l})\psi \right] \right\rangle dt.$$

Using Lemma 1 of Sect. 5.1 we get

$$|A| \leqslant |\!|\!| T |\!|\!|_p \, N_{p'}(\psi) N_{p'}(l) \, I$$

where

$$I = \left(\int_G^* N_p\left(({}_{t^{-1}}(\check{k})\varphi) * (\Delta_G^{1/p'} \alpha) - {}_{t^{-1}}(\check{k})\varphi \right)^p dt \right)^{1/p}.$$

We have

$$I \leqslant N_p(\lambda_1) + N_p(\lambda_2)$$

where for every $t \in G$

$$\lambda_1(t) = \int_G^* \alpha(y) N_p\left(({}_{t^{-1}}(\check{k})\varphi)_{y^{-1}} \Delta_G(y^{-1})^{1/p} - ({}_{t^{-1}}(\check{k})_y \ \varphi)_{y^{-1}} \Delta_G(y^{-1})^{1/p} \right) dy$$

and where

$$\lambda_2(t) = \int_G^* \alpha(y) N_p\left(({}_{t^{-1}}(\check{k})_y \ \varphi)_{y^{-1}} \Delta_G(y^{-1})^{1/p} - {}_{t^{-1}}(\check{k})\varphi \right) dy.$$

To conclude it suffices to check, again by Lemma 1 of Sect. 5.1, that

$$N_p(\lambda_1) \leqslant N_p(\varphi) \sup_{y \in \operatorname{supp} \alpha} N_p(k - {}_{y^{-1}}k)$$

and

$$N_p(\lambda_2) \leqslant N_p(k) \sup_{y \in \operatorname{supp} \alpha} N_p\left(\varphi_{y^{-1}} \Delta_G(y^{-1})^{1/p} - \varphi \right).$$

Lemma 3. *Let G be a locally compact group, $1 < p < \infty$, $k, \varphi \in \mathcal{L}^p(G)$, $l, \psi \in \mathcal{L}^{p'}(G)$ and $T \in CV_p(G)$. Then*

$$\left\langle \left((\overline{k} * \check{l})T \right)[\tau_p \varphi], [\tau_{p'} \psi] \right\rangle = \left\langle \left((\overline{\varphi} * \check{\psi})T \right)[\tau_p k], [\tau_{p'} l] \right\rangle.$$

Proof. Let $k, \varphi, l, \psi, \alpha \in \mathcal{M}_{00}^{\infty}(G)$. By Lemma 1 we have

$$\left\langle \left((\overline{k}*\check{l})(T\lambda_G^p(\overline{\alpha^*}))\right)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle = \left\langle T\left[\alpha\Delta_G^{1/p'}\right], \left[\Delta_G^{-1/p'}\overline{k}*\check{l}\overline{\varphi}*\check{\psi}\right]\right\rangle_{L^p_{\mathbb{C}}(G), L^{p'}_{\mathbb{C}}(G)}$$

and

$$\left\langle \left((\overline{\varphi}*\check{\psi})(T\lambda_G^p(\overline{\alpha^*}))\right)[\tau_p k], [\tau_{p'} l]\right\rangle = \left\langle T\left[\alpha\Delta_G^{1/p'}\right], \left[\Delta_G^{-1/p'}\overline{\varphi}*\check{\psi}\overline{k}*\check{l}\right]\right\rangle.$$

This implies

$$\left\langle \left((\overline{k}*\check{l})(T\lambda_G^p(\overline{\alpha^*}))\right)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle = \left\langle \left((\overline{\varphi}*\check{\psi})(T\lambda_G^p(\overline{\alpha^*}))\right)[\tau_p k], [\tau_{p'} l]\right\rangle.$$

Choosing α like in Lemma 2 and taking its support arbitrary small we conclude.

Lemma 4. *Let G be a locally compact group, $1 < p < \infty$, $\big((k_n), (l_n)\big) \in \mathcal{A}_p(G)$ and $T \in CV_p(G)$. Then the sequence*

$$\left(\sum_{n=1}^{N}(\overline{k}_n * \check{l}_n)T\right)$$

converges in $\mathcal{L}(L^p(G))$ for the operator norm. Let S be the unique element of $\mathcal{L}(L^p(G))$ with

$$\lim_{N\mapsto\infty}\left\|\left\|S - \sum_{n=1}^{N}(\overline{k}_n * \check{l}_n)T\right\|\right\|_p = 0.$$

Then $S \in CV_p(G)$ and

$$|||S|||_p \leqslant |||T|||_p \sum_{n=1}^{\infty} N_p(k_n)N_{p'}(l_n).$$

Proof. Let $1 \leqslant M < N$. We have

$$\left\|\left\|\sum_{n=1}^{N}(\overline{k}_n * \check{l}_n)T - \sum_{n=1}^{M}(\overline{k}_n * \check{l}_n)T\right\|\right\|_p \leqslant |||T|||_p \sum_{n=1+M}^{N} N_p(k_n)N_{p'}(l_n).$$

Consequently there is an unique $S \in \mathcal{L}(L^p(G))$ with

$$\lim_{N\mapsto\infty}\left\|\left\|S - \sum_{n=1}^{N}(\overline{k}_n * \check{l}_n)T\right\|\right\|_p = 0.$$

We have $S \in CV_p(G)$, and for every $N \in \mathbb{N}$

$$\|\!|S|\!\|_p \leqslant \left\|\!\left| S - \sum_{n=1}^{N} (\overline{k_n} * \check{l}_n) T \right|\!\right\|_p + \|\!|T|\!\|_p \sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n).$$

Definition 1. The operator S of Lemma 4 is denoted $\sum_{n=1}^{\infty} (\overline{k_n} * \check{l}_n) T$.

Lemma 5. *Let G be a locally compact group, $1 < p < \infty$ and $\big((k_n),(l_n)\big)$, $\big((k'_n),(l'_n)\big) \in \mathcal{A}_p(G)$ with $\sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n = \sum_{n=1}^{\infty} \overline{k'_n} * \check{l}'_n$. Then for every $T \in CV_p(G)$ we have*

$$\sum_{n=1}^{\infty} (\overline{k_n} * \check{l}_n) T = \sum_{n=1}^{\infty} (\overline{k'_n} * \check{l}'_n) T.$$

Proof. Let $\varphi, \psi \in C_{00}(G)$. It suffices to verify that

$$\left\langle \left(\sum_{n=1}^{\infty} (\overline{k_n} * \check{l}_n) T \right) [\varphi], [\psi] \right\rangle = \left\langle \left(\sum_{n=1}^{\infty} (\overline{k'_n} * \check{l}'_n) T \right) [\varphi], [\psi] \right\rangle.$$

Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ with

$$\sum_{n=1+N}^{\infty} N_p(k_n) N_{p'}(l_n) < \frac{\varepsilon}{6\big(1 + \|\!|T|\!\|_p\big)\big(1 + N_p(\varphi)\big)\big(1 + N_{p'}(\psi)\big)}$$

and

$$\sum_{n=1+N}^{\infty} N_p(k'_n) N_{p'}(l'_n) < \frac{\varepsilon}{6\big(1 + \|\!|T|\!\|_p\big)\big(1 + N_p(\varphi)\big)\big(1 + N_{p'}(\psi)\big)}.$$

There is also U open neighborhood of e such that for every $y \in U$ and for every $1 \leqslant n \leqslant N$

$$N_p(k_n - {}_{y^{-1}}k_n) < \frac{\varepsilon}{12 \cdot 2^n \big(1 + \|\!|T|\!\|_p\big)\big(1 + N_p(\varphi)\big)\big(1 + N_{p'}(\psi)\big)\big(1 + N_{p'}(l_n)\big)},$$

$$N_p\Big(\varphi - \varphi_{y^{-1}} \Delta_G(y^{-1})^{1/p}\Big)$$
$$< \frac{\varepsilon}{12 \cdot 2^n \big(1 + \|\!|T|\!\|_p\big)\big(1 + N_{p'}(\psi)\big)\big(1 + N_{p'}(l_n)\big)\big(1 + N_{p'}(l'_n)\big)\big(1 + N_p(k_n)\big)\big(1 + N_p(k'_n)\big)}$$

and

$$N_p(k'_n -_{y^{-1}} k'_n) < \frac{\varepsilon}{12 \cdot 2^n \big(1 + |||T|||_p\big)\big(1 + N_p(\varphi)\big)\big(1 + N_{p'}(\psi)\big)\big(1 + N_{p'}(l'_n)\big)}.$$

We choose $\alpha \in C_{00}(G)$ with $\alpha \geq 0$, $\int\limits_G \alpha(x)dx = 1$ and $\operatorname{supp}\alpha \subset U$. Lemma 3 implies

$$\left|\left\langle\Big(\sum_{n=1}^{\infty}(\overline{k}_n * \check{l}_n)T\Big)[\varphi],[\psi]\right\rangle - \left\langle\Big(\sum_{n=1}^{\infty}(\overline{k'}_n * \check{l'}_n)T\Big)[\varphi],[\psi]\right\rangle\right|$$
$$< \varepsilon + \left|\sum_{n=1}^{\infty}\left\langle\Big((\overline{k}_n * \check{l}_n)(T\lambda_G^p(\overline{\alpha^*}))\Big)[\varphi],[\psi]\right\rangle - \sum_{n=1}^{\infty}\left\langle\Big((\overline{k'}_n * \check{l'}_n)(T\lambda_G^p(\overline{\alpha^*}))\Big)[\varphi],[\psi]\right\rangle\right|.$$

Now let $u = \sum\limits_{n=1}^{\infty} \overline{k}_n * \check{l}_n$. Using Lemma 1 we get

$$\sum_{n=1}^{\infty}\left\langle\Big((\overline{k}_n * \check{l}_n)(T\lambda_G^p(\overline{\alpha^*}))\Big)[\varphi],[\psi]\right\rangle = \left\langle T\Big[\alpha\Delta_G^{1/p'}\Big], \Big[u\Delta_G^{-1/p'}\overline{\tau_p\varphi} * (\tau_{p'}\psi)\check{}\,\Big]\right\rangle$$

and finally

$$\left|\left\langle\Big(\sum_{n=1}^{\infty}(\overline{k}_n * \check{l}_n)T\Big)[\varphi],[\psi]\right\rangle - \left\langle\Big(\sum_{n=1}^{\infty}(\overline{k'}_n * \check{l'}_n)T\Big)[\varphi],[\psi]\right\rangle\right| < \varepsilon.$$

We now show that the convolution operator $(\overline{\varphi} * \check{\psi})T$ is more regular than T. In fact this operator can be considered as a noncommutative smoothing of the operator T.

Theorem 6. *Let G be a locally compact group, $1 < p < \infty$, $\varphi \in \mathcal{L}^p(G)$, $\psi \in \mathcal{L}^{p'}(G)$ and $T \in CV_p(G)$. Then:*

1. $(\overline{\varphi} * \check{\psi})T \in PM_p(G)$,
2. *for every $u \in A_p(G)$ we have*

$$\Big\langle u, (\overline{\varphi} * \check{\psi})T\Big\rangle_{A_p,PM_p} = \sum_{n=1}^{\infty}\overline{\left\langle\Big((\overline{\varphi} * \check{\psi})T\Big)[\tau_p k_n],[\tau_{p'} l_n]\right\rangle}$$

for every $\big((k_n),(l_n)\big) \in \mathcal{A}_p(G)$ with $u = \sum\limits_{n=1}^{\infty}\overline{k}_n * \check{l}_n$.

Proof. To prove (1) it suffices (see Corollary 8 of Sect. 4.1) to verify that

$$\sum_{n=1}^{\infty}\Big\langle\Big((\overline{\varphi}*\check{\psi})T\Big)[\tau_p k_n],[\tau_{p'} l_n]\Big\rangle=\sum_{n=1}^{\infty}\Big\langle\Big((\overline{\varphi}*\check{\psi})T\Big)[\tau_p k_n'],[\tau_{p'} l_n']\Big\rangle$$

for $\Big((k_n),(l_n)\Big),\Big((k_n'),(l_n')\Big)\in\mathcal{A}_p(G)$ with $\sum_{n=1}^{\infty}\overline{k}_n*\check{l}_n=\sum_{n=1}^{\infty}\overline{k_n'}*\check{l}_n'$.

Lemma 3 implies

$$\sum_{n=1}^{\infty}\Big\langle\Big((\overline{\varphi}*\check{\psi})T\Big)[\tau_p k_n],[\tau_{p'} l_n]\Big\rangle=\sum_{n=1}^{\infty}\Big\langle\Big((\overline{k_n}*\check{l}_n)T\Big)[\tau_p \varphi],[\tau_{p'} \psi]\Big\rangle.$$

But by Lemma 5

$$\sum_{n=1}^{\infty}\Big\langle\Big((\overline{k_n}*\check{l}_n)T\Big)[\tau_p \varphi],[\tau_{p'} \psi]\Big\rangle=\sum_{n=1}^{\infty}\Big\langle\Big((\overline{k_n'}*\check{l}_n')T\Big)[\tau_p \varphi],[\tau_{p'} \psi]\Big\rangle$$

consequently

$$\sum_{n=1}^{\infty}\Big\langle\Big((\overline{\varphi}*\check{\psi})T\Big)[\tau_p k_n],[\tau_{p'} l_n]\Big\rangle=\sum_{n=1}^{\infty}\Big\langle\Big((\overline{\varphi}*\check{\psi})T\Big)[\tau_p k_n'],[\tau_{p'} l_n']\Big\rangle.$$

The point (2) follows from Definition 4 of Sect. 4.1.

Definition 2. Let G be a locally compact group and $1<p<\infty$. For $u\in A_p(G)$ and $T\in CV_p(G)$ we put

$$uT=\sum_{n=1}^{\infty}(\overline{k}_n*\check{l}_n)T$$

where $\Big((k_n),(l_n)\Big)\in\mathcal{A}_p(G)$ with $u=\sum_{n=1}^{\infty}\overline{k}_n*\check{l}_n$.

Remark. It follows from Lemma 4 that $uT\in CV_p(G)$ and that $|||uT|||_p\leqslant|||T|||_p\,\|u\|_{A_p}$.

We can generalize Theorem 6.

Theorem 7. *Let G be a locally compact group, $1<p<\infty$ and $T\in CV_p(G)$. Then:*

1. $uT\in PM_p(G)$ *for every* $u\in A_p(G)$,
2. *for every* $u,v\in A_p(G)$ *we have*

$$\big\langle u,vT\big\rangle_{A_p,PM_p}=\big\langle v,uT\big\rangle_{A_p,PM_p}.$$

Proof. Let $\Big((k_n),(l_n)\Big) \in \mathcal{A}_p(G)$ with $u = \sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n$. By Lemma 4 and Theorem 6 we have $uT \in PM_p(G)$. It remains to prove (2).

Let $\Big((r_n),(s_n)\Big) \in \mathcal{A}_p(G)$ with $v = \sum_{n=1}^{\infty} \overline{r}_n * \check{s}_n$. We have

$$\langle v, uT\rangle_{A_p,PM_p} = \lim_{n\mapsto\infty} \sum_{j=1}^{n} \Big\langle v, (\overline{k}_j * \check{l}_j)T\Big\rangle_{A_p,PM_p}.$$

But for every $j \in \mathbb{N}$

$$\Big\langle v, (\overline{k}_j * \check{l}_j)T\Big\rangle_{A_p,PM_p} = \sum_{m=1}^{\infty} \overline{\Big\langle \Big((\overline{k}_j * \check{l}_j)T\Big)[\tau_p r_m], [\tau_{p'} s_m]\Big\rangle},$$

by Lemma 3

$$\Big\langle v, (\overline{k}_j * \check{l}_j)T\Big\rangle_{A_p,PM_p} = \sum_{m=1}^{\infty} \overline{\Big\langle \Big((\overline{r}_m * \check{s}_m)T\Big)[\tau_p k_j], [\tau_{p'} l_j]\Big\rangle}$$

$$= \sum_{m=1}^{\infty} \Big\langle \overline{k}_j * \check{l}_j, (\overline{r}_m * \check{s}_m)T\Big\rangle_{A_p,PM_p} = \Big\langle \overline{k}_j * \check{l}_j, vT\Big\rangle_{A_p,PM_p}$$

and consequently

$$\langle v, uT\rangle_{A_p,PM_p} = \langle u, vT\rangle_{A_p,PM_p}.$$

Corollary 8. *Let G be a locally compact group, $1 < p < \infty$ and $T \in PM_p(G)$. Then for $u, v \in A_p(G)$ we have*

$$\langle u, vT\rangle_{A_p,PM_p} = \langle uv, T\rangle_{A_p,PM_p}.$$

Proof. For $k, l, r, s \in C_{00}(G)$ we have

$$\Big\langle \overline{k} * \check{l}\, \overline{r} * \check{s}, T\Big\rangle_{A_p,PM_p} = \int_G \Big\langle \overline{r k_h} * (s l_h)\check{}, T\Big\rangle_{A_p,PM_p} dh$$

$$= \int_G \overline{\Big\langle T\Big[{}_{h^{-1}}(\check{k})\tau_p r\Big], \Big[{}_{h^{-1}}(\check{l})\tau_{p'} s\Big]\Big\rangle} dh = \overline{\Big\langle \Big((\overline{k} * \check{l})T\Big)[\tau_p r], [\tau_{p'} s]\Big\rangle}$$

$$= \Big\langle \overline{r} * \check{s}, (\overline{k} * \check{l})T\Big\rangle_{A_p,PM_p}.$$

We then put for every $u \in A_p(G)$

$$F(u) = \left\langle u(\overline{k} * \check{l}), T \right\rangle_{A_p(G), PM_p(G)}.$$

Then $F \in A_p(G)'$. There is therefore an unique $S \in PM_p(G)$ with

$$F(u) = \langle u, S \rangle_{A_p, PM_p}.$$

In particular for every $r, s \in C_{00}(G)$

$$F(\overline{r} * \check{s}) = \left\langle \overline{r} * \check{s}, S \right\rangle_{A_p, PM_p} = \left\langle \overline{r} * \check{s}\, \overline{k} * \check{l}, T \right\rangle_{A_p, PM_p}.$$

This implies

$$(\overline{k} * \check{l})T = S \quad \text{and} \quad \left\langle u, (\overline{k} * \check{l})T \right\rangle_{A_p, PM_p} = \left\langle u(\overline{k} * \check{l}), T \right\rangle_{A_p, PM_p}.$$

Remark. Many authors use the relation $\langle u, vT \rangle_{A_p, PM_p} = \langle uv, T \rangle_{A_p, PM_p}$ to define vT for $T \in PM_p(G)$.

5.3 $CV_p(G)$ as a Module on $A_p(G)$

Lemma 1. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$, $\varphi, \psi \in C_{00}(G)$ and $\varepsilon > 0$. Then there is U open neighborhood of e in G such that*

$$\left| \left\langle (uT)[\varphi], [\psi] \right\rangle - \left\langle \left(u(T \lambda_G^p(\overline{\alpha^*})) \right)[\varphi], [\psi] \right\rangle \right| \leqslant \varepsilon \, |||T|||_p$$

for every $T \in CV_p(G)$ and every $\alpha \in C_{00}(G)$ with $\alpha \geq 0$, $\int_G \alpha(x)dx = 1$ and $\operatorname{supp} \alpha \subset U$.

Proof. Let $\left((k_n), (l_n) \right) \in \mathcal{A}_p(G)$ with $\sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n = u$. There is $N \in \mathbb{N}$ with

$$\sum_{n=1+N}^{\infty} N_p(k_n) N_{p'}(l_n) < \frac{\varepsilon}{4\left(1 + N_p(\varphi)\right)\left(1 + N_{p'}(\psi)\right)}.$$

It suffices (taking in account Lemma 2 of Sect. 5.2) to choose an open neighborhood U of e in G such that:

$$N_p(k_n - {}_{y^{-1}} k_n) < \frac{\varepsilon}{4 \cdot 2^n \big(1 + N_p(\varphi)\big)\big(1 + N_{p'}(\psi)\big)\big(1 + N_{p'}(l_n)\big)}$$

and

$$N_p\left(\varphi - \varphi_{y^{-1}} \Delta_G(y^{-1})^{1/p}\right) < \frac{\varepsilon}{4 \cdot 2^n \big(1 + N_{p'}(\psi)\big)\big(1 + N_{p'}(l_n)\big)\big(1 + N_p(k_n)\big)}$$

for every $y \in U$ and for every $1 \leqslant n \leqslant N$.

Lemma 2. *Let G be a locally compact group,* $1 < p < \infty$, $k, l, r, s, \varphi, \psi \in C_{00}(G)$ *and* $\alpha \in C_{00}(G)$ *with* $\alpha \geq 0$, $\int_G \alpha(x)dx = 1$. *For every* $T \in CV_p(G)$ *we then have*

$$\left|\left\langle\Big((\overline{r} * \check{s})((\overline{k} * \check{l})(T\lambda_G^p(\overline{\alpha^*})))\Big)[\varphi], [\psi]\right\rangle - \left\langle\Big((\overline{r} * \check{s})((\overline{k} * \check{l})T)\Big)[\varphi], [\psi]\right\rangle\right|$$

$$\leqslant \|\!|T|\!\|_p\, N_{p'}(l)N_{p'}(\psi)N_{p'}(s)\Bigg\{N_p(\varphi)N_p(r)\left(\int_G \alpha(y)N_p(k - {}_y k)^p dy\right)^{1/p}$$

$$+N_p(\varphi)N_p(k)\left(\int_G \alpha(y)N_p(r - {}_y r)^p dy\right)^{1/p}$$

$$+N_p(k)N_p(r)\left(\int_G \alpha(y)N_p\Big(\varphi - \varphi_{y^{-1}}\Delta_G(y^{-1})^{1/p}\Big)^p dy\right)^{1/p}\Bigg\}.$$

Proof. Let $U = (\overline{k} * \check{l})(T\lambda_G^p(\overline{\alpha^*}))$ and

$$I = \left\langle\Big((\overline{r} * \check{s})U\Big)[\varphi], [\psi]\right\rangle - \left\langle\Big((\overline{r} * \check{s})\Big((\overline{k} * \check{l})T\Big)\Big)[\varphi], [\psi]\right\rangle.$$

According to Proposition 4 of Sect. 5.1 we have

$$I = \left\langle\Big((\overline{r} * \check{s})\Big(U - (\overline{k} * \check{l})T\Big)\Big)[\varphi], [\psi]\right\rangle$$

and therefore

$$I = \int_G \left\langle \left(U - (\overline{k} * \check{l})T\right)\left[{}_{z^{-1}}(\check{r})\varphi\right], \left[{}_{z^{-1}}(\check{s})\psi\right]\right\rangle dz.$$

For $z, t \in G$ we put

$$\lambda_1(t, z) = \int_G \alpha(y) N_p\left(\left({}_{z^{-1}}(\check{r})\varphi\right)\left({}_{t^{-1}}(\check{k}) -_{t^{-1}} (\check{k})_y\right)\right) dy,$$

$$\lambda_2(t, z) = \int_G \alpha(y) N_p\left((\varphi_{y^{-1}})\left({}_{z^{-1}}(\check{r})_{y^{-1}} -_{z^{-1}} (\check{r})\right)({}_{t^{-1}}(\check{k}))\Delta_G(y)^{-1/p}\right) dy$$

and

$$\lambda_3(t, z) = \int_G \alpha(y) N_p\left({}_{z^{-1}}(\check{r})\left({}_{t^{-1}}(\check{k})\right)\left(\varphi_{y^{-1}}\Delta_G(y)^{-1/p} - \varphi\right)\right) dy.$$

Using Lemma 1 of Sect. 5.1 we get

$$|I| \leqslant \|T\|_p \, N_{p'}(l) N_{p'}(s) N_{p'}(\psi)$$
$$\times \left\{ \left(\iint_{G\times G} \lambda_1(t, z)^p \, dt\, dz \right)^{1/p} + \left(\iint_{G\times G} \lambda_2(t, z)^p \, dt\, dz \right)^{1/p} + \left(\iint_{G\times G} \lambda_3(t, z)^p \, dt\, dz \right)^{1/p} \right\}$$

with

$$\left(\iint_{G\times G} \lambda_1(t, z)^p \, dt\, dz \right)^{1/p} \leqslant N_p(r) N_p(\varphi) \left(\int_G \alpha(y) N_p(k -_{y^{-1}} k)^p \, dy \right)^{1/p},$$

$$\left(\iint_{G\times G} \lambda_2(t, z)^p \, dt\, dz \right)^{1/p} \leqslant N_p(k) N_p(\varphi) \left(\int_G \alpha(y) N_p(r -_y r)^p \, dy \right)^{1/p}$$

and

$$\left(\iint_{G\times G} \lambda_3(t, z)^p \, dt\, dz \right)^{1/p} \leqslant N_p(k) N_p(r) \left(\int_G \alpha(y) N_p\left(\varphi_{y^{-1}}\Delta_G(y^{-1})^{1/p} - \varphi\right)^p dy \right)^{1/p}.$$

We need the following extension of Lemma 1 of Sect. 5.2.

Lemma 3. *Let G be a locally compact group, $1 < p < \infty$, $T \in CV_p(G)$, $u \in A_p(G)$ and $\alpha, \varphi, \psi \in M_{00}^{\infty}(G)$. Then*

$$\left\langle \left(u\left(T\lambda_G^p(\overline{\alpha^*})\right)\right)[\varphi],[\psi]\right\rangle = \left\langle T\left[\alpha\Delta_G^{1/p'}\right], \left[u\Delta_G^{-1/p'}\overline{\tau_p(\varphi)} * (\tau_{p'}(\psi))^{\vee}\right]\right\rangle.$$

Proof. Let $\left((k_n),(l_n)\right) \in \mathcal{A}_p(G)$ with $\sum_{n=1}^{\infty}\overline{k}_n * \check{l}_n = u$. By Lemma 1 of Sect. 5.2 we have

$$\left\langle \left(u\left(T\lambda_G^p(\overline{\alpha^*})\right)\right)[\varphi],[\psi]\right\rangle = \sum_{n=1}^{\infty}\left\langle T\left[\alpha\Delta_G^{1/p'}\right], \left[\overline{k}_n * \check{l}_n\Delta_G^{-1/p'}\overline{\tau_p(\varphi)} * (\tau_{p'}(\psi))^{\vee}\right]\right\rangle_{L_{\mathbb{C}}^p(G),L_{\mathbb{C}}^{p'}(G)}.$$

The uniform convergence of $\sum_{n=1}^{\infty}\overline{k}_n * \check{l}_n$ on G implies that

$$\sum_{n=1}^{\infty}\left\langle T\left[\alpha\Delta_G^{1/p'}\right], \left[\overline{k}_n * \check{l}_n\Delta_G^{-1/p'}\overline{\tau_p(\varphi)} * (\tau_{p'}(\psi))^{\vee}\right]\right\rangle = \left\langle T\left[\alpha\Delta_G^{1/p'}\right], \left[u\,\Delta_G^{-1/p'}\overline{\tau_p(\varphi)} * (\tau_{p'}(\psi))^{\vee}\right]\right\rangle.$$

Theorem 4. *Let G be a locally compact group and $1 < p < \infty$. Then for the mapping $(u,T) \mapsto uT$, $CV_p(G)$ is a left normed $A_p(G)$-module.*

Proof. It remains to show that $u(vT) = (uv)T$ for $u, v \in A_p(G)$ and $T \in CV_p(G)$. Clearly it suffices to verify that for $k, l, r, s \in C_{00}(G)$ one has

$$(\overline{r} * \check{s})\left((\overline{k} * \check{l})T\right) = \left(\overline{r} * \check{s}\,\overline{k} * \check{l}\right)T.$$

Let $\varphi, \psi \in C_{00}(G)$ and $\varepsilon > 0$. By Lemma 1 and 2 there is $\alpha \in C_{00}(G)$ with $\alpha \geq 0$, $\int_G \alpha(y)dy = 1$,

$$\left|\left\langle \left(\left(\overline{r} * \check{s}\,\overline{k} * \check{l}\right)T\right)[\varphi],[\psi]\right\rangle - \left\langle \left(\left(\overline{r} * \check{s}\,\overline{k} * \check{l}\right)\left(T\lambda_G^p(\overline{\alpha^*})\right)\right)[\varphi],[\psi]\right\rangle\right| < \frac{\varepsilon}{2}$$

and

$$\left|\left\langle \left((\overline{r} * \check{s})\left((\overline{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)\right)\right)[\varphi],[\psi]\right\rangle - \left\langle \left((\overline{r} * \check{s})\left((\overline{k} * \check{l})T\right)\right)[\varphi],[\psi]\right\rangle\right| < \frac{\varepsilon}{2}.$$

By Lemma 3

$$\left\langle \left((\overline{r} * \check{s}\,\overline{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)\right)[\varphi],[\psi]\right\rangle_{L_{\mathbb{C}}^p(G),L_{\mathbb{C}}^{p'}(G)} = \left\langle T\left[\alpha\Delta_G^{\frac{1}{p'}}\right], \left[\overline{r} * \check{s}\,\overline{k} * \check{l}\,\Delta_G^{-\frac{1}{p'}}\overline{\tau_p\varphi} * (\tau_{p'}\psi)^{\vee}\right]\right\rangle.$$

Let

$$I = \left\langle \left((\overline{r} * \check{s})\left((\overline{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*}) \right) \right) \right)[\varphi], [\psi] \right\rangle.$$

We have

$$I = \int_G \left\langle \left((\overline{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*}) \right) \right)\left[{}_{t^{-1}}(\check{r})\varphi \right], \left[{}_{t^{-1}}(\check{s})\psi \right] \right\rangle dt$$

and taking account Lemma 1 of Sect. 5.2 we get

$$\begin{aligned} I &= \int_G \left\langle T\left[\alpha\Delta_G^{1/p'} \right], \left[\overline{k} * \check{l}\,\Delta_G^{-1/p'}\overline{\tau_p({}_{t^{-1}}(\check{r})\varphi)} * (\tau_{p'}({}_{t^{-1}}(\check{s})\psi)^{\check{}} \right] \right\rangle dt \\ &= \left\langle T\left[\alpha\Delta_G^{1/p'} \right], \left[\overline{k} * \check{l}\,\Delta_G^{-1/p'} \int_G \overline{\tau_p({}_{t^{-1}}(\check{r})\varphi)} * (\tau_{p'}({}_{t^{-1}}(\check{s})\psi)^{\check{}}\, dt \right] \right\rangle. \end{aligned}$$

Lemma 3 of Sect. 3.3 implies

$$I = \left\langle T\left[\alpha\Delta_G^{1/p'} \right], \left[\overline{k} * \check{l}\,\Delta_G^{-1/p'}\overline{r} * \check{s}\overline{\tau_p\varphi} * (\tau_{p'}(\psi)^{\check{}} \right] \right\rangle$$

thus

$$\left\langle \left((\overline{r} * \check{s})\left((\overline{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*}) \right) \right) \right)[\varphi], [\psi] \right\rangle = \left\langle \left((\overline{r} * \check{s}\,\overline{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*}) \right) \right)[\varphi], [\psi] \right\rangle$$

and finally

$$\left| \left\langle \left((\overline{r} * \check{s}\,\overline{k} * \check{l})T \right)[\varphi], [\psi] \right\rangle - \left\langle \left((\overline{r} * \check{s})\left((\overline{k} * \check{l})T \right) \right)[\varphi], [\psi] \right\rangle \right| < \varepsilon.$$

Remarks. 1. The fact that $CV_p(G)$ is an $A_p(G)$-module is due to Herz (see [61], p. 116). For the preceding approach see Derighetti [29], [30].

2. For G an abelian locally compact group, $u \in A_2(G)$ and $T \in CV_p(G)$ we have

$$(uT)\widehat{\ } = \Phi_{\hat{G}}^{-1}(\overline{u}) * \widehat{T}.$$

Consequently $(uT)\widehat{\ }$ is a smoothing of $\widehat{T}$ in the usual sense.

We finally generalize Proposition 5 of Sect. 5.1.

Proposition 5. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$ and $\mu \in M^1(G)$. Then $u\lambda_G^p(\mu) = \lambda_G^p(\tilde{u}\mu)$.*

Proof. Let $\Big((k_n),(l_n)\Big) \in \mathcal{A}_p(G)$ with $u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n$. According to Proposition 5 of Sect. 5.1

$$u\lambda_G^p(\mu) = \sum_{n=1}^{\infty}\Big(\overline{k_n} * \check{l}_n\Big)\lambda_G^p(\mu) = \sum_{n=1}^{\infty}\lambda_G^p\Big(\big(\overline{k_n} * \check{l}_n\big)\tilde{}\mu\Big) = \lambda_G^p(\tilde{u}\mu).$$

5.4 Approximation of T by $(k * \check{l})T$ for an Amenable G

Lemma 1. *Let G be a locally compact group, $1 < p < \infty$, $k \in \mathcal{L}^p(G)$ and $l \in \mathcal{L}^{p'}(G)$ with $\int_G k(x)\overline{l(x)}dx = 1$. For $T \in CV_p(G)$ and $\varphi, \psi \in C_{00}(G)$ we then have*

$$\Big|\Big\langle\Big((\overline{k} * \check{l})T\Big)[\varphi],[\psi]\Big\rangle - \Big\langle T[\varphi],[\psi]\Big\rangle\Big|$$
$$\leqslant |||T|||_p\, N_p(\varphi)N_{p'}(\psi\Big\{N_{p'}(l)\sup_{y\in\operatorname{supp}\varphi} N_p({}_{y^{-1}}k - k) + N_p(k)\sup_{y\in\operatorname{supp}\psi} N_{p'}({}_{y^{-1}}l - l)\Big\}.$$

Proof. First observe that

$$\Big\langle T[\varphi],[\psi]\Big\rangle = \int_G \Big\langle T[k(t)\varphi],[l(t)\psi]\Big\rangle dt$$

and therefore

$$\Big|\Big\langle\Big((\overline{k} * \check{l})T\Big)[\varphi],[\psi]\Big\rangle - \Big\langle T[\varphi],[\psi]\Big\rangle\Big| \leqslant |||T|||_p \int_G^* N_p\Big({}_{t^{-1}}(\check{k})\varphi - k(t)\varphi\Big)N_{p'}\Big({}_{t^{-1}}(\check{l})\psi\Big)dt$$
$$+ |||T|||_p \int_G^* N_p\Big(k(t)\varphi\Big)N_{p'}\Big({}_{t^{-1}}(\check{l})\psi - l(t)\psi\Big)dt.$$

But

$$\int_G^* N_p\Big(k(t)\varphi\Big)^p dt = N_p(k)^p N_p(\varphi)^p, \left(\int_G^* N_{p'}\Big({}_{t^{-1}}(\check{l})\psi\Big)dt\right)^{1/p'} = N_{p'}(\psi)N_{p'}(l),$$

$$\int_G^* N_p\Big({}_{t^{-1}}(\check{k})\varphi - k(t)\varphi\Big)^p dt = \int_G |\varphi(y)|^p N_p\Big({}_{y^{-1}}k - k\Big)^p dy$$

and

$$\int_G^* N_{p'}\Big({}_{t^{-1}}(\check{l})\psi - l(t)\psi\Big)^{p'} dt = \int_G |\psi(y)|^{p'} N_{p'}\Big({}_{y^{-1}}l - l\Big)^{p'} dy.$$

Let G be an amenable locally compact group and $1 < p < \infty$. For K compact subset of G and $\varepsilon > 0$ there is $f \in C_{00}(G)$ with $f \geqslant 0$, $\int_G f(x)dx = 1$ and

$$N_1\big({}_{y^{-1}}f - f\big) < \min\left\{\left(\frac{\varepsilon}{2}\right)^p, \left(\frac{\varepsilon}{2}\right)^{p'}\right\}$$

for every $y \in K$ (Sect. 1.1 of Chap. 1). Let $k = f^{1/p}$ and $l = f^{1/p'}$, then $k \geq 0$, $l \geq 0$, $N_p(k) = N_{p'}(l) = \int_G k(t)l(t)dt = 1$, $N_p\big({}_{y^{-1}}k - k\big) < \varepsilon$ and $N_{p'}\big({}_{y^{-1}}l - l\big) < \varepsilon$ for every $y \in K$.

Theorem 2. *Let G be an amenable locally compact group, $1 < p < \infty$ and*

$$I = \Big\{(K, \varepsilon, k, l) \Big| K \quad \textit{is a compact subset of } G, 0 < \varepsilon < \infty,$$

$$k, l \in C_{00}(G) \quad \textit{with} \quad N_p(k) = N_{p'}(l) = \int_G k(t)l(t)dt = 1, k \geq 0, l \geq 0$$

$$N_p\big({}_{y^{-1}}k - k\big) < \varepsilon \quad \textit{and} \quad N_{p'}({}_{y^{-1}}l - l) < \varepsilon \quad \textit{for every } y \in K\Big\}.$$

Then:

1. *On I the relation $K \subset K'$ and $\varepsilon \geq \varepsilon'$ is a filtering partial order,*
2. *For every $T \in CV_p(G)$ the net $\Big((k * \check{l})T\Big)_{(K,\varepsilon,k,l) \in I}$ converges ultraweakly to T.*

Proof. For every $(K, \varepsilon, k, l) \in I$ and $T \in CV_p(G)$ we have

$$\Big|\Big|\Big|(k * \check{l})T\Big|\Big|\Big|_p \leq |||T|||_p\,.$$

To prove (2) it suffices therefore to show that

$$\lim_{(K,\varepsilon,k,l)\in I} \left\langle \left((\overline{k} * \check{l})T \right)[\varphi], [\psi] \right\rangle = \left\langle T[\varphi], [\psi] \right\rangle$$

for every $\varphi, \psi \in C_{00}(G)$.

Let $\varepsilon > 0$, $T \in CV_p(G)$, $\varphi, \psi \in C_{00}(G)$, $K_0 = \operatorname{supp}\varphi \cup \operatorname{supp}\psi$ and $k_0, l_0 \in C_{00}(G;\mathbb{R})$ with $(K_0, \eta_0, k_0, l_0) \in I$ where

$$\eta_0 = \frac{\varepsilon}{2(1 + |||T|||_p)(1 + N_p(\varphi))(1 + N_{p'}(\psi))}.$$

Then Lemma 1 implies that for $(K_0, \eta_0, k_0, l_0) \le (K, \eta, k, l)$

$$\left| \left\langle \left((\overline{k} * \check{l})T \right)[\varphi], [\psi] \right\rangle - \left\langle T[\varphi], [\psi] \right\rangle \right| < \varepsilon.$$

Corollary 3. *Let G be an amenable locally compact group and $1 < p < \infty$. Then $CV_p(G) = PM_p(G)$.*

Proof. Let $T \in CV_p(G)$. By Theorem 6 of Sect. 5.2 for $k, l \in C_{00}(G)$ $(\overline{k} * \check{l})T \in PM_p(G)$, finally Theorem 2 permits to conclude.

Remark. In the notes to Chap. 5 we give a list of nonamenable groups with $PM_p(G) = CV_p(G)$.

Theorem 4. *Let G be an amenable locally compact group, $1 < p < \infty$ and I as in Theorem 2. Then for $u \in A_p(G)$ the net $\left((k * \check{l})u\right)_{(K,\varepsilon,k,l)\in I}$ converges to u in $A_p(G)$.*

Proof. Let $\varepsilon > 0$. There is (r_n), (s_n) sequences of $C_{00}(G)$ with

$$\sum_{n=1}^{\infty} N_p(r_n) N_{p'}(s_n) < \|u\|_{A_p(G)} + \varepsilon_1 \quad \text{and} \quad \sum_{n=1}^{\infty} \overline{r_n} * \check{s_n} = u$$

where $0 < \varepsilon_1 < \min\{1, \varepsilon\}$. There is $N \in \mathbb{N}$ with

$$\sum_{n=1+N}^{\infty} N_p(r_n) N_{p'}(s_n) < \frac{\varepsilon}{4}$$

and $k_0, l_0 \in C_{00}(G;\mathbb{R})$ with

$$\left(K_0, \frac{\varepsilon}{4\left(1 + \|u\|_{A_p}\right)}, k_0, l_0 \right) \in I$$

where $\operatorname{supp} \tau_p r_j \cup \operatorname{supp} \tau_p s_j \subset K_0$ for every $1 \le j \le N$.

We claim that

$$\|u - \overline{k} * \check{l}u\|_{A_p} \leq \varepsilon$$

for every $(K, \eta, k, l) \in I$ with

$$\left(K_0, \frac{\varepsilon}{4\big(1 + \|u\|_{A_p}\big)}, k_0, l_0\right) \preceq (K, \eta, k, l)$$

where $\preceq$ is the partial order of Theorem 2.

According to Theorem 6 of Sect. 4.1 it suffices to verify that for every $T \in PM_p(G)$ with $|\!|\!| T |\!|\!|_p \leq 1$

$$\left|\big\langle u - \overline{k} * \check{l}u, T\big\rangle_{A_p, PM_p}\right| < \varepsilon.$$

By Corollary 8 of Sect. 5.2

$$\big\langle \overline{k} * \check{l}u - u, T\big\rangle_{A_p, PM_p} = \big\langle u, (\overline{k} * \check{l})T - T\big\rangle_{A_p, PM_p}.$$

By Lemma 1 for every $1 \leq n \leq N$ we have

$$\left|\big\langle \overline{r_n} * \check{s}_n, (\overline{k} * \check{l})T - T\big\rangle_{A_p, PM_p}\right| \leq \frac{\varepsilon N_p(r_n) N_{p'}(s_n)}{2\big(1 + \|u\|_{A_p}\big)}$$

and therefore

$$\sum_{n=1}^{N} \left|\big\langle \overline{r_n} * \check{s}_n, (\overline{k} * \check{l})T - T\big\rangle_{A_p, PM_p}\right| < \frac{\varepsilon}{2}.$$

For every $n \in \mathbb{N}$

$$\left|\big\langle \overline{r_n} * \check{s}_n, (\overline{k} * \check{l})T - T\big\rangle_{A_p, PM_p}\right| \leq 2 N_p(r_n) N_{p'}(s_n)$$

this implies

$$\left|\sum_{n=N+1}^{\infty} \big\langle \overline{r_n} * \check{s}_n, (\overline{k} * \check{l})T - T\big\rangle_{A_p, PM_p}\right| < \frac{\varepsilon}{2}$$

and finally

$$\left|\big\langle u - \overline{k} * \check{l}u, T\big\rangle_{A_p, PM_p}\right| < \varepsilon.$$

Remarks. 1. It follows that for an amenable G the Banach algebra $A_p(G)$ has an approximate unit in the following sense: for every $u \in A_p(G)$ and for every $\varepsilon > 0$ there is $v \in A_p(G)$ with $\|v\|_{A_p} \leq 1$ and $\|u - uv\|_{A_p} < \varepsilon$. This result was first obtained by Leptin for $p = 2$ [76]. Leptin's proof was based on the following property: for every compact subset K of G and for every $\varepsilon > 0$ there is a compact set U with $m_G(U) > 0$ and $m_G(KU) < (1 + \varepsilon)m_G(U)$. He also proved that there is no hope to obtain such an approximate unit for non amenable groups. See also Pier's book ([100], p. 96).

2. The preceding theorem is directly inspired from the proof of Lemma 5 of [61], p. 121.

Chapter 6
The Support of a Convolution Operator

We define the support of a convolution operator. Using this notion we obtain a full generalization of Wiener's theorem to arbitrary locally compact groups.

6.1 Definition of the Support of a Convolution Operator

Definition 1. Let G be a locally compact group, $1 < p < \infty$ and $T \in CV_p(G)$. We call support of the convolution operator T, the set of all $x \in G$ such that for every neighborhood U of e and for every neighborhood V of x there is $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset U$ $\operatorname{supp}\psi \subset V$ and $\left\langle T[\varphi], [\psi]\right\rangle \neq 0$. The support of T is denoted $\operatorname{supp} T$.

Remark. Clearly $\operatorname{supp} T$ is a closed subset of G.

Proposition 1. *Let G be a locally compact group,* $1 < p < \infty$ *and* $\mu \in M^1(G)$. *Then* $\operatorname{supp}\lambda_G^p(\mu) = (\operatorname{supp}\mu)^{-1}$.

Proof. We show at first that $(\operatorname{supp}\mu)^{-1} \subset \operatorname{supp}\lambda_G^p(\mu)$. Let $x \in G \setminus \operatorname{supp}\lambda_G^p(\mu)$. There is U, V open subsets of G, both relatively compact, with $e \in U$ $x \in V$ and $\left\langle \lambda_G^p(\mu)[\varphi], [\psi]\right\rangle = 0$ for every $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset U$ and $\operatorname{supp}\psi \subset V$. Let $f \in C_{00}(G)$ with $\operatorname{supp} f \subset V$, Z an open neighborhood of e in G with $Z \subset U$ and such that for every $z \in Z$

$$\left\| (\tau_p f)_{z^{-1}} \Delta_G(z^{-1}) - \tau_p f \right\|_u < \frac{\varepsilon}{\left(1 + \|\mu\|_{M^1(G)}\right)\left(\sup_{y \in V^{-1}U} \Delta_G(y)^{1/p}\right)}.$$

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Now choose $\alpha \in C_{00}(G)$ with $\alpha \geqslant 0$, $\operatorname{supp}\alpha \subset Z$ and $\int_G \alpha(y)dy = 1$. From

$$|\check{\mu}(f)| < \varepsilon + \left|\mu\left(\Delta_G^{1/p}(\tau_p f * \alpha)\right)\right| \quad \text{and} \quad \mu\left(\Delta_G^{1/p}(\tau_p f * \alpha)\right) = 0$$

we get $\check{\mu}(f) = 0$ and therefore $x^{-1} \notin (\operatorname{supp}\mu)^{-1}$.

We prove now that $\operatorname{supp}\lambda_G^p(\mu) \subset (\operatorname{supp}\mu)^{-1}$. Let $x \in G \setminus (\operatorname{supp}\mu)^{-1}$. There is W open neighborhood of x^{-1} such that $\mu(\varphi) = 0$ for every $\varphi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset W$. There is also U open neighborhood of e in G and V open neighborhood of x in G with $V^{-1}U \subset W$. Let $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset U$, $\operatorname{supp}\psi \subset V$. Consider $f(y) = \int_G \varphi(xy)\Delta_G(y)^{1/p}\overline{\psi(x)}dx$. From $\mu(f) = \left\langle \lambda_G^p(\mu)[\varphi], [\psi] \right\rangle$ and $\operatorname{supp} f \subset W$ we get $\left\langle \lambda_G^p(\mu)[\varphi], [\psi] \right\rangle = 0$ and therefore $x \notin \operatorname{supp}\lambda_G^p(\mu)$.

Definition 2. Let G be a locally compact abelian group and $u \in L^\infty(\widehat{G})$. We call spectrum of u the set of all $x \in G$ such that $\varepsilon_G(x)^{\cdot}$ belongs to the closure of

$$\left\{ \sum_{j=1}^n c_j \, {}_{\chi_j}u \,\middle|\, n \in \mathbb{N}, c_1, \ldots, c_n \in \mathbb{C}, \chi_1, \ldots, \chi_n \in \widehat{G} \right\}$$

in $L^\infty(\widehat{G})$ with respect to the topology $\sigma(L^\infty, L^1)$. This subset of G is denoted *spu*.

Concerning this important notion we refer to Chap. 7 of [105].

Theorem 2. *Let G be a locally compact abelian group, $u \in L^\infty(\widehat{G})$ and $x \in G$. Then the following properties are equivalent:*

1. $x \in spu$,
2. *For every $f \in L^1(\widehat{G})$ with $f * u = 0$ we have $\hat{f}(\varepsilon_G(x)) = 0$,*
3. *For every $f \in L^1(\widehat{G})$ with $f^* * u = 0$ we have $\hat{f}(\varepsilon_G(x)) = 0$,*
4. *For every open neighborhood W of x there is $h \in L^1(\widehat{G})$ with $\operatorname{supp}(\widehat{h} \circ \varepsilon_G) \subset W$ and $\langle h, u \rangle \neq 0$.*

Proof. Let V_u be the following subspace of $L^\infty_{\mathbb{C}}(\widehat{G})$

$$\left\{ \sum_{j=1}^n c_j \, {}_{\chi_j}u \,\middle|\, n \in \mathbb{N}, c_1, \ldots, c_n \in \mathbb{C}, \chi_1, \ldots, \chi_n \in \widehat{G} \right\}.$$

1. (1) implies (2).

For every $v \in V_u$ we have $\langle f^*, v \rangle = 0$. Let $\eta > 0$. There is $w \in V_u$ such that

$$\left| \left\langle f^*, \varepsilon_G(x)^{\cdot} \right\rangle - \left\langle f^*, w \right\rangle \right| < \eta$$

and therefore

$$\left|\left\langle f^*, \varepsilon_G(x)^\cdot \right\rangle\right| < \eta.$$

Consequently $\widehat{f}(\varepsilon_G(x)) = 0$.

2. (1) implies (3).

The proof is similar to 2.

3. (2) implies (1).

Suppose that $\varepsilon_G(x)^\cdot$ is not in the closure of V_u in $L^\infty(\widehat{G})$ with respect to the topology $\sigma(L^\infty, L^1)$. By the bipolar theorem there is $f \in L^1(\widehat{G})$ with $\left\langle f, \varepsilon_G(x)^\cdot \right\rangle \neq 0$ and $\langle f, v\rangle = 0$ for every $v \in V_u$. This implies $f^* * u = 0$ and therefore $\hat{f}(\varepsilon_G(x)) = 0$. A contradiction.

4. (3) implies (1).

The proof is entirely similar to 3.

5. (3) implies (4).

There is $g \in L^1(\widehat{G})$ with $\widehat{g}(\varepsilon_G(x)) \neq 0$ and $\operatorname{supp}(\widehat{g} \circ \varepsilon_G) \subset W$. We have $\tilde{g} * u \neq 0$. Let $\chi_0 \in \widehat{G}$ with $(\tilde{g} * u)(\chi_0) \neq 0$ and $h = {}_{\chi_0^{-1}} g$. Then $\operatorname{supp}(\widehat{h} \circ \varepsilon_G) \subset W$ and $\langle h, u\rangle \neq 0$.

6. (4) implies (3).

Let $f \in L^1(\widehat{G})$ with $\widehat{f}(\varepsilon_G(x)) \neq 0$. Let W be an open relatively compact neighborhood of x in G with $\widehat{f}(\varepsilon_G(y)) \neq 0$ for every $y \in W$. The theorem of Wiener implies the existence of $g \in L^1(\widehat{G})$ such that $\widehat{f}(\varepsilon_G(y))\widehat{g}(\varepsilon_G(y)) = 1$ for every $y \in W$ ([105], Theorem 6.1.1, p. 169). By assumption there is $h \in L^1(\widehat{G})$ with $\operatorname{supp}(\widehat{h} \circ \varepsilon_G) \subset W$ and $\langle h, u\rangle \neq 0$. The relation $\widehat{h} = \widehat{h}\widehat{g}\widehat{f}$ implies $h = h * g * f$. But $\langle h, u\rangle = \langle h * g, \tilde{f} * u\rangle$ and therefore $\tilde{f} * u \neq 0$.

Remarks. 1. The property (2) has been chosen as a definition of the spectrum for $G = \mathbb{R}$ by Pollard in 1953 [101].

2. Definition 2 is due to Godement ([52], p. 128).

Theorem 3. *Let G be a locally compact abelian group, $1 < p < \infty$ and $T \in CV_p(G)$. Then* $(\operatorname{supp} T)^{-1} = sp\widehat{T}$.

Proof. We show at first that $(\operatorname{supp} T)^{-1} \subset sp\widehat{T}$. Let $x \in G \setminus sp\widehat{T}$. There is W a neighborhood of x such that for every $h \in L^1(\widehat{G})$ with $\operatorname{supp}(\hat{h} \circ \varepsilon_G) \subset W$ we have $\left\langle h, \widehat{T}\right\rangle = 0$. There is U and V open subsets of G with $e \in U$, $x^{-1} \in V$ and $UV^{-1} \subset W$. Consider $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset U$ and $\operatorname{supp}\psi \subset V$. We have $\langle T[\varphi], [\psi]\rangle = \left\langle \widehat{T}\mathcal{F}[\varphi], \mathcal{F}[\psi]\right\rangle$ (see notations and facts from Sect. 1.3). Let $h = \overline{\widehat{\varphi}}\, \widehat{\psi}$, then $h \in L^1(\widehat{G})$, $\widehat{h} \circ \varepsilon_G = \overline{\varphi} * \check{\psi}$, $\operatorname{supp} \widehat{h} \circ \varepsilon_G \subset W$ and therefore $\left\langle h, \widehat{T}\right\rangle = 0$. This implies $\langle T[\varphi], [\psi]\rangle = 0$ and consequently $x^{-1} \notin \operatorname{supp} T$.

It remains to verify that $sp\widehat{T} \subset (\operatorname{supp} T)^{-1}$. Let $x \in G \setminus (\operatorname{supp} T)^{-1}$. There is U, V open relatively compact subsets of G with $e \in U$, $x^{-1} \in V$, $\langle T[\varphi], [\psi] \rangle = 0$ for every $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp} \varphi \subset U$ and $\operatorname{supp} \psi \subset V$. Let $W = V^{-1}$. We show that for every $h \in L^1(\widehat{G})$ with $\operatorname{supp} \widehat{h} \circ \varepsilon_G \subset W$ we have $\langle h, \widehat{T} \rangle = 0$. Let $\eta > 0$. There is K compact subset of $\widehat{G}$ such that

$$\int_{\hat{G} \setminus K} |h(\chi)| d\chi < \frac{\eta}{3(1 + \|\widehat{T}\|_\infty)}.$$

There is $r \in L^1(G)$ with $\hat{r} \in C_{00}(\widehat{G})$ and $\widehat{r} = 1$ on K. There is also $\varphi \in C_{00}(G)$ with $\varphi \geqslant 0$, $\int_G \varphi(x) dx = 1$, $\operatorname{supp} \varphi \subset U$ and

$$\|\varphi * r - r\|_1 < \frac{\eta}{3\big(1 + \|h\|_1\big)\big(1 + \|\widehat{T}\|_\infty\big)}.$$

We therefore obtain for every $\chi \in K$

$$|\widehat{\varphi}(\chi) - 1| < \frac{\eta}{3\big(1 + \|h\|_1\big)\big(1 + \|\widehat{T}\|_\infty\big)}.$$

This implies

$$\left\| h - \overline{\widehat{\varphi}} h \right\|_1 < \frac{\eta}{\big(1 + \|\widehat{T}\|_\infty\big)} \quad \text{and} \quad \left| \langle h - \overline{\widehat{\varphi}} h, \widehat{T} \rangle \right| < \eta,$$

from

$$\int_{\hat{G}} \widehat{T}(\chi) \widehat{\varphi}(\chi) \overline{h(\chi)} d\chi = 0 \quad \text{we have} \quad \left| \langle h, \widehat{T} \rangle \right| < \eta$$

and therefore $\langle h, \widehat{T} \rangle = 0$. Theorem 2 finally implies that $x \notin sp\widehat{T}$.

6.2 Characterization of the Support of a Pseudomeasure

In analogy with the property (4) of Theorem 2 of Sect. 6.1, we get at first a characterization of the support of a pseudomeasure.

Theorem 1. *Let G be a locally compact group, $1 < p < \infty$, $T \in PM_p(G)$ and $x \in G$. Then $x \in \operatorname{supp} T$ if and only if for every neighborhood V of x there is $u \in A_p(G)$ with $\operatorname{supp} u \subset V$ and $\langle u, T \rangle_{A_p(G), PM_p(G)} \neq 0$.*

Proof. Suppose that $x \in \operatorname{supp} T$. There is U_1, V_1 open subsets of G with $e \in U_1$, $x \in V_1$, and $U_1^{-1} V_1 \subset V$. There is also $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset U_1$, $\operatorname{supp}\psi \subset V_1$ and $\big\langle T[\varphi], [\psi]\big\rangle \neq 0$. For $u = \overline{(\tau_p\varphi)} * (\tau_p\psi)$ we have $\big\langle u, T\big\rangle_{A_p,PM_p} \neq 0$ with $\operatorname{supp} u \subset V$.

Suppose conversely that for every neighborhood V of x there is $u \in A_p(G)$ with $\operatorname{supp} u \subset V$ and $\big\langle u, T\big\rangle_{A_p,PM_p} \neq 0$. Let U, V open subsets of G with $e \in U$ and $x \in V$. Let W an open neighborhood of x relatively compact with $\overline{W} \subset V$. Choose $u \in A_p(G)$ with $\operatorname{supp} u \subset W$ $\big\langle u, T\big\rangle_{A_p,PM_p} \neq 0$ and $\Big((k_n), (l_n)\Big) \in \mathcal{A}_p(G)$ with $u = \sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n$. There is $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} N_p(k_n) N_{p'}(l_n) < \frac{\varepsilon}{4} \quad \text{where} \quad \varepsilon = \frac{\Big|\big\langle u, T\big\rangle_{A_p,PM_p}\Big|}{2\,|||T|||_p}.$$

There is $\varphi \in C_{00}(G)$ with $\varphi \geqslant 0$, $\int_G \varphi(x)dx = 1$, $\operatorname{supp}\varphi \subset U^{-1}$ and

$$N_p\big(\varphi * k_n - k_n\big) < \frac{\varepsilon}{2 \cdot 2^n\big(1 + N_{p'}(l_n)\big)}$$

for every $1 \leqslant n \leqslant N$. Then $\Big((\varphi * k_n), (l_n)\Big) \in \mathcal{A}_p(G)$ and

$$\left\| u - \sum_{n=1}^{\infty} \overline{(\varphi * k_n)} * \check{l}_n \right\|_{A_p} \leqslant \sum_{n=1}^{N} \frac{\varepsilon N_{p'}(l_n)}{2 \cdot 2^n (1 + N_{p'}(l_n))} + 2 \sum_{n=N+1}^{\infty} N_p(k_n) N_{p'}(l_n) < \varepsilon.$$

From

$$\varphi * u = \sum_{n=1}^{\infty} \overline{(\varphi * k_n)} * \check{l}_n$$

we get

$$\|u - \varphi * u\|_{A_p} < \varepsilon \quad \text{and therefore} \quad \big\langle \varphi * u, T\big\rangle_{A_p,PM_p} \neq 0.$$

This finally implies $\Big\langle T[\tau_p\varphi], [\tau_{p'}\check{u}]\Big\rangle \neq 0$ with $\operatorname{supp} \tau_p\varphi \subset U$ and $\operatorname{supp} \tau_{p'}\check{u} \subset V$.

Theorem 2. *Let G be a locally compact group and $1 < p < \infty$. Then for $u \in A_p(G)$ and $T \in CV_p(G)$ we have $\Big\{x \Big| x \in \operatorname{supp} T, u(x) \neq 0\Big\} \subset \operatorname{supp} uT \subset \operatorname{supp} T \cap \operatorname{supp} u$.*

Proof. 1. Let $x \in G \setminus \operatorname{supp} uT$ with $u(x) \neq 0$. We prove that $x \notin \operatorname{supp} T$.

Choose U and V open relatively compact subsets of G with $e \in U$, $x \in V$ and $\Big\langle (uT)[\varphi], [\psi] \Big\rangle = 0$ for every $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp} \varphi \subset U$ and $\operatorname{supp} \psi \subset V$. There is V_1, open neighborhood of x, with $u(y) \neq 0$ for $y \in V_1$. There is also U_1, V_2 open subsets of G with $e \in U_1$, $x \in V_2$ and $U_1^{-1} V_2 \subset V_1 \cap V$. Let $U_2 = U_1 \cap U$ and $V_3 = V \cap V_2$. Let $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp} \varphi \subset U_2$ and $\operatorname{supp} \psi \subset V_3$. We show that $\langle T[\varphi], [\psi] \rangle = 0$.

Let $\varepsilon > 0$. There is U_3 open neighborhood of e in G such that for every $\alpha \in C_{00}(G)$ with $\alpha \geqslant 0$, $\int_G \alpha(y) dy = 1$, $\operatorname{supp} \alpha \subset U_3$ we have

$$\Big| \langle T \lambda_G^p (\overline{\alpha^*})[\varphi], [\psi] \rangle - \langle T[\varphi], [\psi] \rangle \Big| < \frac{\varepsilon}{2}.$$

According to Corollary 6 of Sect. 4.3 (and Proposition 1 of Sect. 3.1) there is $v \in A_p(G) \cap C_{00}(G)$ with $v(y) = \dfrac{1}{u(y)}$ for every $y \in U_2^{-1} V_3$. There is U_4 open neighborhood of e such that for every $\alpha \in C_{00}(G)$ with $\alpha \geq 0$, $\int_G \alpha(y) dy = 1$ and $\operatorname{supp} \alpha \subset U_4$ we have (see Lemma 1 of Sect. 5.3)

$$\left| \Big\langle (uvT)[\varphi], [\psi] \Big\rangle - \left\langle \Big(uv \Big(T \lambda_G^p (\alpha^*) \Big) \Big) [\varphi], [\psi] \right\rangle \right| < \frac{\varepsilon}{2}.$$

Let $\alpha \in C_{00}(G)$ with $\alpha \geq 0$, $\int_G \alpha(y) dy = 1$ and $\operatorname{supp} \alpha \subset U_3 \cap U_4$. Taking into account Lemma 3 of Sect. 5.3 and

$$uv(\overline{\tau_p \varphi} * (\tau_{p'} \psi)\check{\ }) = \overline{\tau_p \varphi} * (\tau_{p'} \psi)\check{\ },$$

we obtain

$$\left\langle \Big(uv \Big(T \lambda_G^p (\alpha^*) \Big) \Big) [\varphi], [\psi] \right\rangle = \left\langle T \Big[\Delta_G^{1/p'} \alpha \Big], \Big[\Delta_G^{-1/p'} \overline{\tau_p \varphi} * (\tau_{p'} \psi)\check{\ } \Big] \right\rangle = \left\langle T \Big[\varphi * \Delta_G^{1/p'} \alpha \Big], \Big[\psi \Big] \right\rangle.$$

Choose (k_n) and (l_n) in $C_{00}(G)$ with $\Big((k_n), (l_n) \Big) \in \mathcal{A}_p(G)$ and $\sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n = v$. For every $n \in \mathbb{N}$ and $t \in G$ we have

$$\left\langle (uT) \Big[{}_{t^{-1}}(\check{k}_n) \varphi \Big], \Big[{}_{t^{-1}}(\check{l}_n) \psi \Big] \right\rangle = 0$$

but

$$\left\langle \Big((\overline{k}_n * \check{l}_n)(uT) \Big) [\varphi], [\psi] \right\rangle = \int_G \left\langle (uT) \Big[{}_{t^{-1}}(\check{k}_n) \varphi \Big], \Big[{}_{t^{-1}}(\check{l}_n) \psi \Big] \right\rangle dt$$

and therefore $\left\langle (uvT)[\varphi], [\psi] \right\rangle = 0$. This implies

$$\left|\left\langle T[\varphi], [\psi] \right\rangle\right| \le \left|\left\langle T[\varphi], [\psi] \right\rangle - \left\langle T\lambda_G^p(\alpha^*)[\varphi], [\psi] \right\rangle\right| + \left|\left\langle T\lambda_G^p(\alpha^*)[\varphi], [\psi] \right\rangle \right.$$
$$\left. - \left\langle \left(uv\left(T\lambda_G^p(\alpha^*)\right)\right)[\varphi], [\psi] \right\rangle\right| + \left|\left\langle \left(uv\left(T\lambda_G^p(\alpha^*)\right)\right)[\varphi], [\psi] \right\rangle \right.$$
$$\left. - \left\langle (uvT)[\varphi], [\psi] \right\rangle\right| + \left|\left\langle (uvT)[\varphi], [\psi] \right\rangle\right| < \varepsilon$$

and finally $\left\langle T[\varphi], [\psi] \right\rangle = 0$.

2. $\operatorname{supp} uT \subset \operatorname{supp} u$.

Let $x \in \operatorname{supp} uT$. Suppose that $x \notin \operatorname{supp} u$. We can find U open neighborhood of x in G with $U \cap \operatorname{supp} u = \emptyset$. There is $v \in A_p(G)$ with $v(x) = 1$ and $\operatorname{supp} v \subset U$. We have $x \in \operatorname{supp} vuT$. But $uvT = 0$ and therefore $\operatorname{supp} uvT = \emptyset$.

3. $\operatorname{supp}(uT) \subset \operatorname{supp} T$.

Let $x \in G \setminus \operatorname{supp} T$. We choose U, open neighborhood of e and V open neighborhood of x with $\langle T[\varphi], [\psi] \rangle = 0$ for every $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp} \varphi \subset U$ and $\operatorname{supp} \psi \subset V$. Let $\left((k_n), (l_n)\right) \in \mathcal{A}_p(G)$ with $\sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n = u$ and $k_n, l_n \in C_{00}(G)$. For every $n \in \mathbb{N}$ we have

$$\left\langle \left((\overline{k}_n * \check{l}_n)T\right)[\varphi], [\psi] \right\rangle = 0$$

and therefore $\left\langle (uT)[\varphi], [\psi] \right\rangle = 0$, consequently $x \notin \operatorname{supp} uT$.

Corollary 3. *Let G be an amenable locally compact group and $1 < p < \infty$. For every $T \in CV_p(G)$ there is a net (T_α) of $CV_p(G)$ with $\operatorname{supp} T_\alpha$ compact for every α and $\lim T_\alpha = T$ ultraweakly.*

Proof. This is a consequence of Theorem 2 of Sect. 5.4 and of Theorem 2.

6.3 A Generalization of Wiener's Theorem

The following lemma follows easily from [7]

Lemma 1. *Let G be a locally compact group. Then:*

1. *For $f \in \mathbb{C}^G$ locally m_G-integrable, m_G-moderated and such that the measure fm_G has compact support, we have $f \in \mathcal{L}^1(G)$;*

2. *For* $f \in L^p(G)$ *with* $1 < p < \infty$ *such that the measure* fm_G *has compact support, we have* $f \in L^1(G)$*;*
3. *For* $1 \le p < \infty$ *and* $f \in L^p(G)$*, we have* $f = 0$ *if and only if* $\operatorname{supp} f m_G = \emptyset$*;*
4. *For* $r \in C(G)$*,* $1 \le p < \infty$ *and* $f \in \mathcal{L}^p(G)$ *such that* fm_G *has compact support we have* $rf \in \mathcal{L}^p(G)$*.*

Lemma 2. *Let* G *be a locally compact group,* $1 < p < \infty$*,* $T \in CV_p(G)$ *and* K *a compact subset of* G *with* $K \cap \operatorname{supp} T = \emptyset$*. There is* U *and* V *open subsets of* G *with* $e \in U$*,* $K \subset V$ *and* $\langle T[\varphi], [\psi] \rangle = 0$ *for every* $\varphi, \psi \in C_{00}(G)$ *with* $\operatorname{supp} \varphi \subset U$*,* $\operatorname{supp} \psi \subset V$*.*

Proof. For every $k \in K$ there is $U_{(k)}$ open neighborhood of e in G and $V_{(k)}$ open neighborhood of k in G with $\langle T[\varphi], [\psi] \rangle = 0$ for every $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp} \varphi \subset U_{(k)}$, $\operatorname{supp} \psi \subset V_{(k)}$. There is also $W_{(k)}$ open neighborhood of k in G with $\overline{W}_{(k)}$ compact and $\overline{W}_{(k)} \subset V_{(k)}$. There is $k_1, \ldots, k_n \in K$ with $K \subset \bigcup_{j=1}^n W_{(k_j)}$. Let

$$U = \bigcap_{j=1}^n U_{(k_j)} \quad \text{and} \quad V = \bigcup_{j=1}^n W_{(k_j)}.$$

Consider $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp} \varphi \subset U$ and $\operatorname{supp} \psi \subset V$. There is $\tau_1, \ldots, \tau_n \in C(G)$ with $0 \leqslant \tau_j \leqslant 1$, $\tau_j(G \setminus W_{(k_j)}) = 0$ for every $1 \leqslant j \leqslant n$ and $\sum_{j=1}^n \tau_j(x) = 1$ for every $x \in \operatorname{supp} \psi$. We have

$$\psi = \sum_{j=1}^n \tau_j \psi, \ \langle T[\varphi], [\tau_j \psi] \rangle = 0 \quad \text{for every} \quad 1 \leqslant j \leqslant n$$

and finally $\langle T[\varphi], [\psi] \rangle = 0$.

Theorem 3. *Let* G *be a locally compact group,* $1 < p < \infty$ *and* $T \in CV_p(G)$*. Then* $T = 0$ *if and only if* $\operatorname{supp} T = \emptyset$*.*

Proof. Clearly if $T = 0$ then $\operatorname{supp} T = \emptyset$. Suppose conversely that $\operatorname{supp} T = \emptyset$. We show that $T[\varphi] = 0$ for $\varphi \in C_{00}(G)$. By Lemma 1, 3) it suffices to prove that $\operatorname{supp}(T[\varphi]) m_G = \emptyset$. Let $x \in G$. We have $((\operatorname{supp} \varphi)^{-1} x) \cap \operatorname{supp} T = \emptyset$. By Lemma 2 there is U and V open sets of G with $e \in U$, $(\operatorname{supp} \varphi)^{-1} x \subset V$ and $\langle T[r], [s] \rangle = 0$ for every $r, s \in C_{00}(G)$ with $\operatorname{supp} r \subset U$ and $\operatorname{supp} s \subset V$. There is also W open subset of G with $x \in W$ and $(\operatorname{supp} \varphi)^{-1} W \subset V$. Let $k \in C_{00}(G)$ with $\operatorname{supp} k \subset W$. Let $\varepsilon > 0$. There is $g \in C_{00}(G)$ with

$$N_p(\varphi - \varphi * g) < \frac{\varepsilon}{\left(1 + \|\!|T|\!\|_p\right)\left(1 + N_{p'}(k)\right)}$$

and $\operatorname{supp} g \subset U$. We have

$$\Big((T[\varphi]) m_G\Big)(k) = \Big\langle T\Big[\varphi\Big], \Big[\overline{k}\Big]\Big\rangle.$$

But

$$\left|\left\langle T\left[\varphi\right],\left[\overline{k}\right]\right\rangle\right| \leqslant \|T\|_p\, N_p(\varphi-\varphi*g)N_{p'}(k)+\left|\left\langle T\left[\varphi*g\right],\left[\overline{k}\right]\right\rangle\right|$$
$$<\varepsilon+\left|\left\langle T\left[g\right],\left[\varphi^**\overline{k}\right]\right\rangle\right|.$$

From $\operatorname{supp}\varphi^* * \overline{k} \subset (\operatorname{supp}\varphi)^{-1}W \subset V$ we get

$$\left\langle T\left[\varphi\right],\left[\varphi^**\overline{k}\right]\right\rangle=0,\ \left|\left\langle T\left[\varphi\right],\left[\overline{k}\right]\right\rangle\right|<\varepsilon \quad\text{and}\quad \Big(\big(T[\varphi]\big)m_G\Big)(k)=0.$$

This implies $x \notin \operatorname{supp}(T[\varphi])m_G$ and finally $\operatorname{supp}\big(T[\varphi]\big)m_G = \emptyset$.

Remark. For $p = 2$ and G abelian, this result is precisely the dual formulation of Wiener's theorem (see Theorem 3, Sect. 6.1 and [105], Proposition 7.1.9, p. 196).

Proposition 4. *Let G be a locally compact group, $1 < p < \infty$, $T \in CV_p(G)$ and $u \in A_p(G)$. Suppose that $u = 0$ on a neighborhood of* $\operatorname{supp} T$. *Then $uT = 0$.*

Proof. We have $\operatorname{supp} T \cap \operatorname{supp} u = \emptyset$, $\operatorname{supp} uT = \emptyset$ (Theorem 2 of Sect. 6.2) and therefore $uT = 0$.

Proposition 5. *Let G be a locally compact group, $1 < p < \infty$, $T \in CV_p(G)$ with compact support and $u \in A_p(G)$. Suppose that $u = 1$ on a neighborhood of* $\operatorname{supp} T$. *Then $uT = T$.*

Proof. Let v be an arbitrary element of $A_p(G)$. By Proposition 4 $(uv - v)T = 0$ and therefore $v(uT - T) = 0$. Theorem 2 of Sect. 6.2 implies $\operatorname{supp}(uT - T) = \emptyset$ and thus $uT - T = 0$.

Corollary 6. *Let G be a locally compact group, $1 < p < \infty$ and $T \in CV_p(G)$ with compact support. Then $T \in PM_p(G)$.*

Proof. Let U be a compact neighborhood of $\operatorname{supp} T$ and $u \in A_p(G) \cap C_{00}(G)$ with $u(x) = 1$ for every $x \in U$. We have $T = uT$, but according to Theorem 7 of Sect. 5.2, uT is a p-pseudomeasure.

Proposition 7. *Let G be a locally compact group, $1 < p < \infty$, $T \in PM_p(G)$ and $u \in A_p(G) \cap C_{00}(G)$ such that* $\operatorname{supp} u \cap \operatorname{supp} T = \emptyset$. *Then $\langle u, T\rangle_{A_p,PM_p} = 0$.*

Proof. Let U be an open neighborhood of $\operatorname{supp} u$ such that $U \cap \operatorname{supp} T = \emptyset$. According to Proposition 1 of Sect. 3.1 there is $v \in A_p(G) \cap C_{00}(G)$ with $v = 1$ on $\operatorname{supp} u$ and $\operatorname{supp} v \subset U$. Then (using Corollary 8 of Sect. 5.2)

$$\langle u, T\rangle_{A_p,PM_p} = \langle uv, T\rangle_{A_p,PM_p} = \langle u, vT\rangle_{A_p,PM_p},$$

Theorem 2 of Sect. 6.2 and Theorem 3 above imply $vT = 0$, consequently

$$\langle u, T\rangle_{A_p, PM_p} = 0.$$

We also obtain the following characterization of the support of a convolution operator.

Theorem 8. *Let G be a locally compact group, $p > 1$, $T \in CV_p(G)$ and $x \in G$. The $x \in \operatorname{supp} T$ if and only if for every $u \in A_p(G)$ with $uT = 0$ we have $u(x) = 0$.*

Proof. Let $x \in \operatorname{supp} T$ and $u \in A_p(G)$ with $uT = 0$. Suppose that $u(x) \neq 0$. By the Theorem 2 of Sect. 6.2 we have $x \in \operatorname{supp} uT$ and therefore $uT \neq 0$. Consequently $u(x) = 0$.

Conversely suppose that for every $u \in A_p(G)$ with $uT = 0$ we have $u(x) = 0$. Assume that $x \notin \operatorname{supp} T$. Let U be an open relatively compact neighborhood of x with $U \cap \operatorname{supp} T = \emptyset$. Choose $v \in A_p(G)$ with $v(x) = 1$ and $\operatorname{supp} v \subset U$. Then $\operatorname{supp} vT = \emptyset$ and therefore $vT = 0$. This implies $u(x) = 0$.

Remarks. 1. This generalizes the property (2) of Theorem 2, Sect. 6.1.
2. For G a general locally compact group and $p = 2$ this result is due to Eymard (see [41], p. 225 (4.4) Proposition).
3. For $1 < p < \infty$ see Herz [61], p. 119.

6.4 An Approximation Theorem

Lemma 1. *Let G be a locally compact group, $1 < p < \infty$, $\varphi \in L^p(G)$ and $T \in CV_p(G)$. Suppose that $\operatorname{supp} \varphi m_G$ is compact. Then $\operatorname{supp}(T\varphi)m_G \subset \operatorname{supp} \varphi m_G \operatorname{supp} T$.*

Proof. Let $x \notin \operatorname{supp} \varphi m_G \operatorname{supp} T$. By Lemma 2 of Sect. 6.3 there is U and V open subsets of G with $e \in U$, $(\operatorname{supp} \varphi m_G)^{-1} x \subset V$ and $\langle T[r], [s]\rangle = 0$ for every $r, s \in C_{00}(G)$ with $\operatorname{supp} r \subset U$, $\operatorname{supp} s \subset V$. There is Z open neighborhood of e with $(\operatorname{supp} \varphi m_G)^{-1} x Z \subset V$. Let $f \in C_{00}(G)$ with $\operatorname{supp} f \subset W$ where $W = xZ$. Let $\varepsilon > 0$. There is $g \in C_{00}(G)$ with $\operatorname{supp} g \subset U$ and

$$\|\varphi - \varphi * [g]\|_p < \frac{\varepsilon}{\left(1 + |||T|||_p\right)\left(1 + N_{p'}(f)\right)}.$$

According to Lemma 1 of Sect. 6.3 we have $\varphi \in L^1(G)$ and therefore

$$\left\langle T(\varphi * [g]), \left[\overline{f}\right]\right\rangle = \left\langle T\left[g\right], \varphi^* * \left[\overline{f}\right]\right\rangle$$

and

$$\left|\left\langle T(\varphi), \left[\overline{f}\right]\right\rangle\right| < \varepsilon + \left|\left\langle T(\varphi * \left[g\right]), \left[\overline{f}\right]\right\rangle\right|.$$

There is $r \in C_{00}(G)$ with $\left[r\right] = \varphi^* * \left[\overline{f}\right]$. We have $\operatorname{supp} r \subset V$ and therefore

$$\left\langle T\left(\varphi * \left[g\right]\right), \left[\overline{f}\right]\right\rangle = 0.$$

This implies

$$\left|\left\langle T(\varphi), \left[\overline{f}\right]\right\rangle\right| < \varepsilon \quad \text{and consequently} \quad \left((T\varphi)m_G\right)(f) = 0.$$

We finally obtain that $x \notin \operatorname{supp}(T\varphi)m_G$.

Lemma 2. *Let* G *be a locally compact group,* $1 < p < \infty$*,* $f \in C_{00}(G)$ *and* $T \in CV_p(G)$*. Suppose that* $\operatorname{supp} T$ *is compact. Then*

1. $\tau_p T[f] \in L^1_{\mathbb{C}}(G)$,
2. $T\lambda^p_G(\tau_p f) = \lambda^p_G\left(\tau_p T[f]\right)$,
3. $\left\|\left|\lambda^p_G\left(\tau_p T[\Delta_G^{1/p'} f]\right)\right|\right\|_p \leq \|\|T\|\|_p\, N_1(f)$,
4. $\operatorname{supp} \lambda^p_G\left(\tau_p T[f]\right) \subset \operatorname{supp} f \operatorname{supp} T$,
5. $T[f * f] \in C_{00}(G)$.

Proof. To verify (2) and (3) observe that for $\varphi \in C_{00}(G)$

$$\lambda^p_G\left(\tau_p T[\Delta_G^{1/p'} f]\right)[\varphi] = [\varphi] * T[\Delta_G^{1/p'} f] = T\left([\varphi] * [\Delta_G^{1/p'} f]\right)$$

but $[\varphi] * [\Delta_G^{1/p'} f] = \lambda^p_G\left(\overline{f^*}\right)[\varphi]$ consequently $\lambda^p_G\left(\tau_p T[\Delta_G^{1/p'} f]\right) = T\lambda^p_G\left(\overline{f^*}\right)$ this indeed implies (2). We also have

$$\left\|\lambda^p_G\left(\tau_p T[\Delta_G^{1/p'} f]\right)[\varphi]\right\|_p \leq \|\|T\|\|_p\, \|[\varphi] * [\Delta_G^{1/p'} f]\|_p \leq \|\|T\|\|_p\, N_p(\varphi) N_1(f)$$

and therefore (3).

As a direct consequence of Proposition 2 of Sect. 2.1, and of the preceding lemma, we obtain the following approximation theorem.

Theorem 3. *Let* G *be a locally compact group,* $1 < p < \infty$ *and*

$$I = \left\{f \in C_{00}(G) \middle| f \geqslant 0, f(e) \neq 0, \int_G \Delta_G(x)^{-1/p'} f(x)dx = 1\right\}.$$

Consider on I the filtering partial order $f \preceq f'$ defined by $\operatorname{supp} f \supset \operatorname{supp} f'$. *Then for every, $T \in CV_p(G)$ with compact support*

1. *For every $f \in I$ we have*

$$\left\| \lambda_G^p \left(\tau_p T[f * f] \right) \right\|_p \leqslant \| T \|_p ;$$

2. *The net*

$$\left(\lambda_G^p \left(\tau_p T[f * f] \right) \right)_{f \in I}$$

converges strongly to T;

3. *For every $\varepsilon > 0$, for every $\varphi \in L^p(G)$ and for every neighborhood U of* $\operatorname{supp} T$ *there is $f_0 \in I$ such that for every $f \in I$ with $f_0 \preceq f$ we have*

$$\left\| T\varphi - \lambda_G^p \left(\tau_p T[f * f] \right) \varphi \right\|_p < \varepsilon$$

and $\operatorname{supp} \lambda_G^p \left(\tau_p T[f * f] \right) \subset U$.

Remark. Theorem 3 is due to Herz ([61], Proposition 9, p. 117).

Corollary 4. *Let G be a locally compact group, $1 < p < \infty$, $T \in CV_p(G)$ with compact support and U a neighborhood of* $\operatorname{supp} T$. *Then there is a net (f_α) of $C_{00}(G)$ such that:*

1. $\lim \lambda_G^p(f_\alpha) = T$ *ultraweakly,*
2. $\left\| \lambda_G^p(f_\alpha) \right\|_p \leq \| T \|_p$ *for every α,*
3. $\operatorname{supp} f_\alpha \subset U$ *for every α.*

6.5 Application to Amenable Groups

From Corollary 7 of Sect. 4.1 and from Corollary 3 of Sect. 5.4 we know that every $T \in CV_p(G)$ is in the ultraweak closure of $\lambda_G^p(C_{00}(G))$ if G is amenable. In this paragraph we will obtain a more precise result.

Lemma 1. *Let G be a locally compact group, $1 < p < \infty$, U a neighborhood of e, $\varepsilon > 0$ and $\varphi \in L^p(G)$. Then there is V, a neighborhood of e contained in U, such that for $f \in C_{00}(G)$ and $T \in CV_p(G)$ with $f \geq 0$ $\int_G f(x) \Delta_G(x)^{-1/p'} dx = 1$,* $\operatorname{supp} f \subset V$ *one has*

$$\| T\varphi - T\lambda_G^p(\tau_p f)\varphi \|_p \leq \| T \| \, \varepsilon.$$

Proof. A minor change to the proof of Proposition 2 of Sect. 2.1 justifies this lemma.

The following lemma is a consequence of Sects. 5.4 and 6.2.

Lemma 2. *Let G be a locally compact amenable group, $1 < p < \infty$, $T \in CV_p(G)$, U a neighborhood of* $\operatorname{supp} T$, *V a neighborhood of e, K a compact subset of G and $\varepsilon > 0$. Then there is $k, l, f \in C_{00}(G)$ with*

1. $N_p(k) = N_{p'}(l) = \int_G k(x)l(x)dx = 1, k \geq 0, l \geq 0,$
2. $N_p({}_{x^{-1}}k - k) < \varepsilon$ *and* $N_{p'}({}_{x^{-1}}l - l) < \varepsilon$ *for every* $x \in K$;
3. $f \geq 0,\ f(e) \neq 0,\ \int_G f(x)\Delta_G(x)^{-1/p'}dx = 1$;
4. $\operatorname{supp} f \subset V$ *and* $(\operatorname{supp} f)^2 \operatorname{supp}(k * \check{l})T \subset U$.

Definition 1. Let G be a locally compact amenable group, $1 < p < \infty$, $T \in CV_p(G)$ and U a neighborhood of $\operatorname{supp} T$. We define J_U as the set of all 5-tuples $(K, \varepsilon, k, l, f)$ K compact subset of G, $\varepsilon > 0$, $k, l, f \in C_{00}(G)$ with following properties:

$$N_p(k) = N_{p'}(l) = \int_G k(x)l(x)dx = 1, k \geq 0, l \geq 0$$

$$N_p({}_{x^{-1}}k - k) < \varepsilon, N_{p'}({}_{x^{-1}}l - l) < \varepsilon \quad \text{for every} \quad x \in K,$$

$$f \geq 0, f(e) \neq 0, \int_G f(x)\Delta_G(x)^{-1/p'}dx = 1, (\operatorname{supp} f)^2 \operatorname{supp}(k * \check{l})T \subset U.$$

Definition 2. Let G be a locally compact amenable group, $1 < p < \infty$, $T \in CV_p(G)$, U a neighborhood of $\operatorname{supp} T$ and $(K, \varepsilon, k, l, f), (K', \varepsilon', k,' l', f') \in J_U$. We say that $(K, \varepsilon, k, l, f) \preceq (K', \varepsilon', k,' l', f')$ if $K \subset K'$, $\varepsilon \geq \varepsilon'$ and $\operatorname{supp} f \supset \operatorname{supp} f'$.

Theorem 3. *Let G be a locally compact amenable group, $1 < p < \infty$, $T \in CV_p(G)$ and U a neighborhood of* $\operatorname{supp} T$. *Then*

1. *$\preceq$ is a filtering partial order on the set J_U;*
2. *For every $(K, \varepsilon, k, l, f) \in J_U$ we have*

$$\left\|\!\left| \lambda_G^p\Big(\tau_p\big((k * \check{l})T[f * f]\big)\Big) \right|\!\right\|_p \leqslant |\!|\!| T |\!|\!|_p;$$

3. $\operatorname{supp} \lambda_G^p\Big(\tau_p\big((k * \check{l})T[f * f]\big)\Big) \subset U$ *for every* $(K, \varepsilon, k, l, f) \in J_U$;
4. *In the ultraweak topology we have*

$$\lim_{(K,\varepsilon,k,l,f)\in J_U} \lambda_G^p\bigg(\tau_p\Big((k * \check{l})T[f * f]\Big)\bigg) = T.$$

Proof. The claim (1) is a consequence of Lemma 2, while (2) and (3) result from Theorem 3 of the preceding paragraph.

To prove (4) it's enough to show that

$$\lim_{(K,\varepsilon,k,l,f)\in J_U}\left\langle \lambda_G^p\left(\tau_p\left((k*\check{l})T[f*f]\right)\right)[\varphi],[\psi]\right\rangle = \langle T[\varphi],[\psi]\rangle$$

for $\varphi,\psi\in C_{00}(G)$.

By Theorem 2 of Sect. 5.4 for every $\varepsilon>0$ there is K_0 compact subset of G and $\eta_0>0$ with the following property: let K be a compact subset of G with $K_0\subset K$, let $0<\eta\le\eta_0$ and let $k,l\in C_{00}(G)$ with

$$N_p(k)=N_{p'}(l)=\int_G k(x)l(x)dx=1, k\ge 0, l\ge 0, N_p({}_{x^{-1}}k-k)<\eta,\quad N_{p'}({}_{x^{-1}}l-l)<\eta$$

for every $x\in K$, then we have

$$\left|\langle(k*\check{l})T[\varphi],[\psi]\rangle-\langle T[\varphi],[\psi]\rangle\right|<\frac{\varepsilon}{2}.$$

By Lemma 1 there is a neighborhood V of e such that for every $f\in C_{00}(G)$ with $\operatorname{supp} f\subset V$, $f\ge 0$ and $\int_G f(x)\Delta_G(x)^{-1/p'}dx=1$ we have

$$\left\|\left(k*\check{l})T\right)[\varphi]-\left((k*\check{l})T\right)\lambda_G^p(\tau_p f)[\varphi]\right\|_p<\frac{\varepsilon}{2(1+N_{p'}(\psi))}$$

for every $k,l\in C_{00}(G)$ with $N_p(k)=N_{p'}(l)=1, k\ge 0, l\ge 0, \int k(x)l(x)\,dx=1$. By Lemma 2 there is $k_0,l_0,f_0\in C_{00}(G)$ such that $(K_0,\eta_0,k_0,l_0,f_0)\in J_U$. Then for every $(K,\eta,k,l,f)\in J_U$ with $(K_0,\eta_0,k_0,l_0,f_0)\preceq(K,\eta,k,l,f)$ we clearly have

$$\left|\left\langle\lambda_G^p\left(\tau_p\left((k*\check{l})T[f*f]\right)\right)[\varphi],[\psi]\right\rangle-\langle T[\varphi],[\psi]\rangle\right|<\varepsilon.$$

Corollary 4. *Let G be a locally compact amenable group, $1<p<\infty$, $T\in CV_p(G)$ and U a neighborhood of* supp T. *Then there is a net $(f_\alpha)_{\alpha\in J}$ of $C_{00}(G)$ such that:*

1. $\lim\lambda_G^p(f_\alpha)=T$ *ultraweakly,*
2. $\left\|\lambda_G^p(f_\alpha)\right\|_p\le\|T\|_p$ *for every α,*
3. supp $f_\alpha\subset U$ *for every α.*

Proof. Let J_U be the set of Definition 1 with the partial order of Definition 2. For $\alpha = (K, \varepsilon, k, l, f) \in J_U$ we put $f_\alpha = \tau_p\Big((k * \check{l})T[f * f]\Big)$. According to Lemma 2 of Sect. 6.4 we have precisely $f_\alpha \in C_{00}(G)$.

Scholium 5. *Let G be a discrete amenable group,* $1 < p < \infty$ *and* $T \in CV_p(G)$. *Then there is a net* (f_α) *of* $C_{00}(G)$ *such that:*

1. $\lim \lambda_G^p(f_\alpha) = T$ *ultraweakly,*
2. $\left|\!\left|\!\left| \lambda_G^p(f_\alpha) \right|\!\right|\!\right|_p \leq |\!|\!| T |\!|\!|_p$ *for every* α,
3. $\operatorname{supp} f_\alpha \subset \operatorname{supp} T$ *for every* α.

Remarks. 1. Corollary 4 (and Corollary 4 of Sect. 6.4) were obtained by Herz [61], p. 117.

2. There is no hope to extend the Scholium 5 to non-discrete groups! In fact let G be a non-discrete locally compact abelian group, then there is always a $T \in CV_2(G)$ which is not an ultraweak limit of a net $\left(\lambda_G^2(\mu_\alpha)\right)$ with μ_α a measure with finite support contained in $\operatorname{supp} T$ [91].
3. For results on the approximation of $T \in CV_p(G)$ by finitely supported measures see Lohoué [80], Théorème II, p. 82, [79], Corollaire 2, [81], Théorème 1, Meyer [94], p. 665, Delmonico [25], Corollary 3.3 and [26], Derighetti [33], Theorem 9.

Chapter 7
Convolution Operators Supported by Subgroups

Let H be a closed subgroup of G. We investigate the relations between $CV_p(H)$ and $CV_p(G)$ and obtain noncommutative analogs of the relations between $L^\infty(\widehat{H})$ and $L^\infty(\widehat{G})$. We prove that $Res_H A_p(G) = A_p(H)$. We also generalize to noncommutative groups the Theorem of Kaplansky–Helson and basic results on sets of synthesis.

7.1 The Image of a Convolution Operator

Let G be a locally compact group and H a closed subgroup. For $x \in G$ we put $\dot{x} = xH = \omega(x)$ and $a \cdot \dot{x} = (ax)^{\cdot}$ for $a \in G$.

For the following we refer to Chap. 8 of [105].

There is $q \in C(G)$ with $q(x) > 0$, $q(xh) = q(x)\Delta_H(h)\Delta_G(h)^{-1}$ for every $x \in G$ and every $h \in H$. Let m_H be a left invariant Haar measure of the locally compact group H. We set $m_H(f) = \int_H f(h)dh$ for $f \in C_{00}(H)$.

Definition 1. For $f \in C_{00}(G)$ and $x \in G$ we put

$$T_{H,q}f(\dot{x}) = \int_H \frac{f(xh)}{q(xh)}dh$$

and

$$T_H f(\dot{x}) = \int_H f(xh)dh.$$

Theorem 1. *There is a unique Radon measure μ on the locally compact space G/H, such that*

$$m_G(f) = \mu(T_{H,q}f)$$

for every $f \in C_{00}(G)$.

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Remark. The measure μ depends on the choice of q.

Definition 2. The unique measure μ of Theorem 1 is denoted $m_{G/H}$. For $f \in C_{00}(G/H)$ we put

$$m_{G/H}(f) = \int\limits_{G/H} f(\dot{x}) d_q \dot{x}.$$

We also put

$$m_{G/H} = \frac{m_G}{m_H}.$$

If $q = 1_G$ (for instance if H is normal in G or if H is compact) we set $d_q \dot{x} = d\dot{x}$.

Theorem 2. *Let G be a locally compact group, H a closed subgroup, ω the canonical map of G onto G/H, $q \in C(G)$ with $q(x) > 0$ $q(xh) = q(x)\Delta_H(h)\Delta_G(h)^{-1}$ for every $x \in G$ and every $h \in H$, m_G a left invariant Haar measure on G, m_H a left invariant Haar measure on H and $m_{G/H}$ such that*

$$m_{G/H} = \frac{m_G}{m_H}.$$

Then for every $f \in \mathcal{L}^1(G)$ there is a unique $g \in L^1(G/H; m_{G/H})$ such that $\omega(f m_G) = g m_{G/H}$. If $f \in C_{00}(G)$ we have $g = \left[T_{H,q} f\right]$.

This permits us to complete Definition 1.

Definition 3. We put $g = T_{H,q}[f]$.

Theorem 3. 1. *$T_{H,q}$ is a contractive epimorphism of the Banach space $L^1(G)$ onto the Banach space $L^1(G/H; m_{G/H})$,*
2. *if H is normal in G then T_H is contractive epimorphism of the involutive Banach algebra $L^1(G)$ onto the involutive Banach algebra $L^1(G/H)$.*

Definition 4. For every $x, y \in G$ we put $\chi(x, \dot{y}) = \dfrac{q(xy)}{q(y)}$.

Proposition 4. *For every $f \in C_{00}(G/H)$ and every $y \in G$ we have*

$$\int\limits_{G/H} f(\dot{x}) d_q \dot{x} = \int\limits_{G/H} f(y \cdot \dot{x}) \chi(y, \dot{x}) d_q \dot{x}.$$

Corollary 5. *Let G be a locally compact group and H a closed normal subgroup of G. Then $m_{G/H}$ is a left invariant Haar measure of the locally compact group G/H.*

Definition 5. Let G be a locally compact group and H a closed subgroup of G. We say that $\beta \in C(G)$ is a Bruhat function for H if:

i. for every compact subset K of G there is $r \in C_{00}(G)$ with $r(x) \geq 0$ for every $x \in G$ and $Res_{KH}\beta = Res_{KH}r$,

ii. $\int\limits_H \beta(xh)dh = 1$ for every $x \in G$.

Theorem 6. *Let G be a locally compact group and H a closed subgroup of G. Then there is a Bruhat function for H.*

Proposition 7. *Let G be a locally compact group, H a closed subgroup of G and β a Bruhat function for H. If we put*

$$q(x) = \int\limits_H \beta(xh)\Delta_G(h)\Delta_H(h^{-1})dh, x \in G$$

we have:

1. $q \in C(G)$*,*
2. $q(x) > 0$ *for* $x \in G$*,*
3. $q(xh) = q(x)\Delta_H(h)\Delta_G(h^{-1})$ *for* $x \in G$ *and for* $h \in H$*.*

Lemma 8. *Let G be a locally compact group, H a closed subgroup of G, $p \geq 1$, $f \in C_{00}(G)$ and $\varepsilon > 0$. There is U open neighborhood of e in G such that*

$$\left(\int\limits_H |f(y^{-1}xh) - f(xh)|^p dh\right)^{1/p} < \varepsilon$$

for every $y \in U$ and for every $x \in G$.

Proof. There is $g \in C_{00}(G)$ with $g \geq 0$, $g(x) = 1$ for $x \in V \operatorname{supp} f$ where V is some compact neighborhood of e in G. Then for $x \in G$ and $y \in V$ we have

$$|f(y^{-1}x) - f(x)|^p \leq |f(y^{-1}x) - f(x)|^p g(x).$$

It suffices to choose U, open neighborhood of e in G, with $U^{-1} = U$, $U \subset V$ and

$$|f(y^{-1}z) - f(z)| < \frac{\varepsilon}{\left(1 + \max_{\dot{x} \in G/H} T_H g(\dot{x})\right)^{1/p}}$$

for $y \in U$ and $z \in G$.

Definition 6. Let G be a locally compact group, H a closed subgroup of G, Z a set and $x \in G$. For $f \in Z^G$, we define the map $f_{x,H}$ of H into Z by $f_{x,H}(h) = f(xh)$ for every $h \in H$. Observe that $f_{e,H} = Res_H f$.

By a straightforward application of Lemma 8 we obtain:

Lemma 9. *Let G be a locally compact group, H a closed subgroup of G, $p > 1$, $\varphi, \psi \in C_{00}(G)$ and $\varepsilon > 0$. Then there is U open neighborhood of e in G such that*

$$\left|\left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle - \left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{y^{-1}x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{y^{-1}x,H}\right]\right\rangle\right| \leq \varepsilon \, |||T|||_p$$

for every $y \in U$, for every $x \in G$ and for every $T \in CV_p(H)$.

Proposition 10. *Let G be a locally compact group, H a closed subgroup of G, $p > 1$, $\varphi, \psi \in C_{00}(G)$ and $\mu \in M^1(H)$. The function*

$$\dot{x} \mapsto \left\langle \lambda_H^p(\mu)\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle$$

is well defined and continuous with compact support on G/H. Let $i(\mu)$ be the image of the measure μ under the inclusion map i of H in G. Then

$$\langle \lambda_G^p(i(\mu))[\varphi], [\psi]\rangle = \int\limits_{G/H} \left\langle \lambda_H^p(\mu)\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle d_q\dot{x}.$$

Proof. We have by Sect. 1.2

$$\langle \lambda_G^p(i(\mu))[\varphi], [\psi]\rangle = \int\limits_G \left(\varphi * \Delta_G^{1/p'}(i(\mu))\check{\ }\right)(x)\overline{\psi(x)}dx$$

where $\left(\varphi * \Delta_G^{1/p'}(i(\mu))\check{\ }\right)\overline{\psi} \in C_{00}(G)$ and therefore

$$\int\limits_G \left(\varphi * \Delta_G^{1/p'}(i(\mu))\check{\ }\right)(x)\overline{\psi(x)}dx = \int\limits_{G/H} \left(\int\limits_H \frac{\left(\varphi * \Delta_G^{1/p'}(i(\mu))\check{\ }\right)(xh)\overline{\psi(xh)}}{q(xh)} dh\right) d_q\dot{x}.$$

But for every $x \in G$ and every $h \in H$ we have

$$\left(\varphi * \Delta_G^{1/p'}(i(\mu))\check{\ }\right)(xh) = \int\limits_G \varphi(xhy)\Delta_G(y)^{1/p} d(i(\mu))(y) = \int\limits_H \varphi(xh\eta)\Delta_G(\eta)^{1/p} d\mu(\eta)$$

and therefore

$$\frac{\left(\varphi * \Delta_G^{1/p'}(i(\mu))\check{\ }\right)(xh)}{q(xh)^{1/p}} = \left(\left(\frac{\varphi}{q^{1/p}}\right)_{x,H} * \Delta_H^{1/p'}\check{\mu}\right)(h).$$

This finally implies

$$\int_H \frac{\big(\varphi * \Delta_G^{1/p'}(i(\mu))\check{\ }\big)(xh)\overline{\psi(xh)}}{q(xh)}dh = \int_H \left(\left(\frac{\varphi}{q^{1/p}}\right)_{x,H} * \Delta_H^{1/p'}\check{\mu}\right)(h)\overline{\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}(h)}dh$$
$$= \left\langle \lambda_H^p(\mu)\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle.$$

Lemma 11. *Let G be a locally compact group, H a closed subgroup of G, $p > 1$, $\varphi, \psi \in C_{00}(G)$ and $T \in CV_p(H)$. Then*

$$\left|\int_{G/H} \left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle d_q\dot{x}\right| \leq |||T|||_p\, N_p(\varphi)N_{p'}(\psi).$$

Proof. We have

$$\left|\int_{G/H} \left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle d_q\dot{x}\right|$$
$$\leq \int_{G/H}^{*} |||T|||_p\, N_p\left(\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right)N_{p'}\left(\left(\frac{\varphi}{q^{1/p'}}\right)_{x,H}\right)d_q\dot{x}$$
$$\leq |||T|||_p\left(\int_{G/H}^{*} N_p\left(\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right)^p d_q\dot{x}\right)^{1/p}\left(\int_{G/H}^{*} N_{p'}\left(\left(\frac{\varphi}{q^{1/p'}}\right)_{x,H}\right)^{p'} d_q\dot{x}\right)^{1/p'}.$$

But

$$\int_{G/H}^{*} N_p\left(\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right)^p d_q\dot{x} = \int_{G/H}^{*}\left(\int_H \frac{|\varphi(xh)|^p}{q(xh)}dh\right)d_q\dot{x} = N_p(\varphi)^p.$$

Proposition 12. *Let G be a locally compact group, H a closed subgroup of G, $p > 1$, and $T \in CV_p(H)$. Then there is an unique bounded operator S of $L^p(G)$ such that*

$$\langle S[\varphi], [\psi]\rangle = \int_{G/H} \left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle d_q\dot{x}$$

for every $\varphi, \psi \in C_{00}(G)$. We have $|||S|||_p \leq |||T|||_p$.

Definition 7. The operator S of Proposition 12 is called the image of T under the map i. We set $S = i(T)$.

Remark. According to Proposition 10 we have $i(\lambda_H^p(\mu)) = \lambda_G^p(i(\mu))$ for $\mu \in M^1(H)$.

Theorem 13. *Let G be a locally compact group, H a closed subgroup of G and $p > 1$. Then i is a linear contractive map of the Banach space $CV_p(H)$ into $CV_p(G)$.*

Proof. It is enough to prove that $i(CV_p(H)) \subset CV_p(G)$. Let $T \in CV_p(H), \varphi, \psi \in C_{00}(G), a \in G$ and

$$f(\dot{x}) = \left\langle T\left[\left(\frac{{}_a\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle.$$

By Proposition 4 we have

$$\left\langle i(T)\,{}_a[\varphi], [\psi]\right\rangle = \int\limits_{G/H} \chi(a^{-1}, \dot{x}) f(a^{-1}\cdot\dot{x}) d_q\dot{x}.$$

From

$$f(a^{-1}\cdot\dot{x}) = \left\langle T\left[\left(\frac{{}_a\varphi}{q^{1/p}}\right)_{a^{-1}x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{a^{-1}x,H}\right]\right\rangle,$$

$$\left(\frac{{}_a\varphi}{q^{1/p}}\right)_{a^{-1}x,H} = \left(\frac{q(x)}{q(a^{-1}x)}\right)^{1/p}\left(\frac{\varphi}{q^{1/p}}\right)_{x,H} \quad \text{and}$$

$$\left(\frac{\psi}{q^{1/p'}}\right)_{a^{-1}x,H} = \left(\frac{q(x)}{q(a^{-1}x)}\right)^{1/p'}\left(\frac{{}_{a^{-1}}\psi}{q^{1/p'}}\right)_{x,H}$$

we get

$$f(a^{-1}\cdot\dot{x}) = \frac{1}{\chi(a^{-1},\dot{x})}\left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{{}_{a^{-1}}\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle$$

and therefore

$$\begin{aligned}\left\langle i(T)\,{}_a[\varphi], [\psi]\right\rangle &= \int\limits_{G/H} \left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{{}_{a^{-1}}\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle d_q\dot{x} \\ &= \left\langle i(T)[\varphi], [{}_{a^{-1}}\psi]\right\rangle = \left\langle {}_a\left(i(T)[\varphi]\right), [\psi]\right\rangle.\end{aligned}$$

7.2 On the Operator $i(T)$

The relation

$$\big\langle i(T)[\varphi],[\psi]\big\rangle = \int_{G/H} \left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right],\left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle d_q\dot{x}$$

was obtained in the preceding paragraph for $\varphi,\psi \in C_{00}(G)$. We need a generalization to $\varphi \in \mathcal{L}^p(G)$ and $\psi \in \mathcal{L}^{p'}(G)$.

Definition 1. Let X be a topological space. Then $\mathcal{T}^+(X)$ denotes the set

$$\left\{f \in [0,\infty]^X \,\middle|\, f \quad \text{is lower semi-continuous on } X\right\}.$$

Proposition 1. *Let G be a locally compact group, H a closed subgroup of G and $f \in \mathcal{T}^+(G)$. Then:*

1. $\dot{x} \mapsto \overset{*}{\int_H} \dfrac{f(xh)}{q(xh)}dh$ *is in* $\mathcal{T}^+(G/H)$,
2. $\overset{*}{\int_{G/H}} \left(\overset{*}{\int_H} \dfrac{f(xh)}{q(xh)}dh\right)d_q\dot{x} = \overset{*}{\int_G} f(x)dx.$

Proof. Let $A = \{\varphi|\varphi \in C_{00}^+(G), \varphi \le f\}$. We recall that $f = \sup\{\varphi|\varphi \in A\}$, for $x \in G$ we have consequently

$$\left(\frac{f}{q}\right)_{x,H} = \sup_{\varphi\in A}\left(\frac{\varphi}{q}\right)_{x,H} \quad \text{and therefore} \quad \left(\frac{f}{q}\right)_{x,H} \in \mathcal{T}^+(H).$$

The set

$$\left\{\left(\frac{\varphi}{q}\right)_{x,H}\middle|\, \varphi \in A\right\}$$

being filtering we have

$$m_H^*\left(\left(\frac{f}{q}\right)_{x,H}\right) = \sup_{\varphi\in A}(T_{H,q}\varphi)(\dot{x}).$$

Let

$$F(\dot{x}) = \overset{*}{\int_H} \frac{f(xh)}{q(xh)}dh$$

we have

$$F = \sup_{\varphi \in A} T_{H,q}\varphi,\ F \in \mathcal{T}^+(G/H) \quad \text{and finally} \quad m^*_{G/H}(F) = \sup_{\varphi \in A} m(\varphi) = m^*(f).$$

Remark. See Reiter and Stegeman ([105]. see p. 230).

Corollary 2. *Let G be a locally compact group, H a closed subgroup of G and f an arbitrary map of G into* $[0, \infty]$. *Then:*

$$\int^*_{G/H} m^*_H\left(\left(\frac{f}{q}\right)_{x,H}\right) d_q\dot{x} \le m^*_G(f).$$

Proof. Let $g \in \mathcal{T}^+(G)$ with $f \le g$. By Proposition 1

$$\int^*_{G/H} m^*_H\left(\left(\frac{f}{q}\right)_{x,H}\right) d_q\dot{x} \le m^*_G(g).$$

This implies

$$\int^*_{G/H} m^*_H\left(\left(\frac{f}{q}\right)_{x,H}\right) d_q\dot{x} \le \inf\left\{m^*_G(g) \,\middle|\, g \in \mathcal{T}^+(G),\, f \le g\right\} = m^*_G(f).$$

The following theorem is a generalization of Theorem 1 of Sect. 7.1.

Theorem 3. *Let G be a locally compact group, H a closed subgroup of G and* $f \in \mathcal{L}^1(G)$. *Then there is a* $m_{G/H}$*-negligible subset A of G/H with the following properties:*

1. *For every* $x \in G$ *with* $\dot{x} \notin A$ *we have* $\left(\frac{f}{q}\right)_{x,H} \in \mathcal{L}^1(H; m_H)$,
2. *The function g, defined by* $g(\dot{x}) = 0$ *if* $\dot{x} \in A$ *and by*

$$g(\dot{x}) = m_H\left(\left(\frac{f}{q}\right)_{x,H}\right)$$

if $\dot{x} \notin A$, *belongs to* $\mathcal{L}^1(G/H)$ *and we have* $m_{G/H}(g) = m_G(f)$.

Proof. There is (k_n) a sequence of $C_{00}(G)$ with

$$\int_G |f(x) - k_n(x)| dx < \frac{1}{2^n}$$

for every $n \in \mathbb{N}$. Corollary 2 implies that

$$\int_{G/H}^{*} \left(\int_{H}^{*} \frac{|f(xh) - k_n(xh)|}{q(xh)} dh \right) d_q \dot{x} < \frac{1}{2^n}$$

for $n \in \mathbb{N}$. We set for every $x \in G$ and for every $n \in \mathbb{N}$

$$H_n(\dot{x}) = \int_{H}^{*} \frac{|f(xh) - k_n(xh)|}{q(xh)} dh,$$

we have $H_n \in [0, \infty]^{G/H}$. Then

$$m^*_{G/H}\left(\sum_{n=1}^{\infty} H_n \right) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

There is A, $m_{G/H}$-negligible subset of G/H, such that for every $\dot{x} \notin A$

$$\sum_{n=1}^{\infty} H_n(\dot{x}) < \infty.$$

For $\dot{x} \notin A$ we have

$$\int_{H}^{*} \frac{|f(xh|}{q(xh)} dh < \infty \quad \text{and} \quad \lim_{n \to \infty} \int_{H}^{*} \left| \left(\frac{f}{q} \right)_{x,H} (h) - \left(\frac{k_n}{q} \right)_{x,H} (h) \right| dh = 0,$$

this implies $\left(\frac{f}{q} \right)_{x,H} \in \mathcal{L}^1(H)$.

Let $g \in \mathbb{R}^{G/H}$ defined by $g(\dot{x}) = 0$ if $\dot{x} \in A$ and by

$$g(\dot{x}) = \int_{H} \frac{f(xh)}{q(xh)} dh$$

if $\dot{x} \notin A$. For $\dot{x} \notin A$ we have

$$|g(\dot{x}) - T_{H,q} k_n(\dot{x})| \leq H_n(\dot{x})$$

and therefore

$$m^*_{G/H}\big(|g - T_{H,q} k_n|\big) \leq m^*_{G/H}(H_n) < \frac{1}{2^n}.$$

This implies that $g \in \mathcal{L}^1(G/H)$. Finally for $n \in \mathbb{N}$ we have

$$\left|m_{G/H}(g) - \int\limits_G f(x)dx\right| \leq \left|m_{G/H}(g) - m_{G/H}\left(T_{H,q}k_n\right)\right|$$
$$+\left|m_{G/H}\left(T_{H,q}k_n\right) - m_G(k_n)\right| + |m_G(k_n) - m_G(f)|$$
$$\leq m_{G/H}\left(|g - T_{H,q}k_n|\right) + m_G(|k_n - f|) < \frac{1}{2^{n-1}}.$$

Remarks. 1. See [105], Theorem 3.4.6, p. 100 and p. 231.
2. Even for $G = \mathbb{R}$ and $H = \mathbb{Z}$ it is not possible in general to choose $A = \emptyset$.

Scholium 4. *Let G be a locally compact group, H a closed subgroup of G, $f \in \mathcal{L}^1(G)$ and A an arbitrary $m_{G/H}$-negligible set such that*

$$\left(\frac{f}{q}\right)_{x,H} \in \mathcal{L}^1(H)$$

for every $\dot{x} \notin A$. Let l be the function defined by $l(\dot{x}) = 0$ if $\dot{x} \in A$ and by

$$l(\dot{x}) = m_H\left(\left(\frac{f}{q}\right)_{x,H}\right)$$

if $\dot{x} \notin A$. Then l belongs to $\mathcal{L}^1(G/H)$ and we have $m_{G/H}(l) = m_G(f)$.

Definition 1. Let G be a locally compact group, H a closed subgroup of G and $f \in \mathcal{L}^1(G)$. A subset A of G/H is said to be associated to f if the following holds:

1. A is $m_{G/H}$-negligible,
2. $$\left(\frac{f}{q}\right)_{x,H} \in \mathcal{L}^1(H)$$
   for every $\dot{x} \notin A$.

Theorem 5. *Le G be a locally compact group, H a closed subgroup of G, $1 < p < \infty$, $f \in \mathcal{L}^p(G)$, $g \in \mathcal{L}^{p'}(G)$, A associated to $|f|^p$, B associated to $|g|^{p'}$ and $T \in CV_p(G)$. For $x \in G$ we put $\lambda(\dot{x}) = 0$ if $\dot{x} \in A \cup B$ and*

$$\lambda(\dot{x}) = \left\langle T\left[\left(\frac{f}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{g}{q^{1/p'}}\right)_{x,H}\right]\right\rangle$$

if $\dot{x} \notin A \cup B$. Then:

1. $\lambda \in \mathcal{L}^1(G/H)$,
2. $m_{G/H}(\lambda) = \langle i(T)[f], [g] \rangle$.

Proof. There are $(f_n), (g_n)$ sequences of $C_{00}(G)$ such that

$$N_p(f - f_n) < \frac{1}{n}$$

and

$$N_{p'}(g - g_n) < \frac{1}{n}.$$

Let

$$\lambda_n(\dot{x}) = \left\langle T\left[\left(\frac{f_n}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{g_n}{q^{1/p'}}\right)_{x,H}\right]\right\rangle$$

for every $x \in G$. According to Proposition 10 of Sect. 7.1 we have $\lambda_n \in C_{00}(G/H)$.

Let $X = (G/H) \setminus (A \cup B)$. For $x \in G$ and $\dot{x} = \omega(x)$ we set

$$r_n(\dot{x}) = \left(\int_H \frac{|f_n(xh)|^p}{q(xh)} dh\right)^{1/p}, \quad s_n(\dot{x}) = \left(\int_H \frac{|g_n(xh) - g(xh)|^{p'}}{q(xh)} dh\right)^{1/p'},$$

$$t_n(\dot{x}) = \left(\int_H \frac{|f_n(xh) - f(xh)|^p}{q(xh)} dh\right)^{1/p}, \quad u(\dot{x}) = \left(\int_H \frac{|g(xh)|^{p'}}{q(xh)} dh\right)^{1/p'}$$

and $r_n(\dot{x}) = s_n(\dot{x}) = t_n(\dot{x}) = u(\dot{x}) = 0$ if $\dot{x} \in A \cup B$. From

$$1_X |\lambda - \lambda_n| \le \|T\|_p \left(1_X r_n s_n + 1_X t_n u\right)$$

we get

$$m^*_{G/H}(|\lambda - \lambda_n|) = m^*_{G/H}(1_X|\lambda - \lambda_n|) \le \|T\|_p \left\{m^*_{G/H}(1_X r_n s_n) + m^*_{G/H}(1_X t_n u)\right\}$$

$$\le \|T\|_p \left\{m^*_{G/H}(1_X r_n^p)^{1/p} m^*_{G/H}(1_X s_n^{p'})^{1/p'} + m^*_{G/H}(1_X t_n^p)^{1/p} m^*_{G/H}(1_X u^{p'})^{1/p'}\right\}.$$

But

$$m^*_{G/H}(1_X r_n^p) = N_p(f_n)^p,$$

$$m^*_{G/H}(1_X s_n^{p'}) = \int^*_{G/H} 1_X(\dot{x}) s_n(\dot{x})^{p'} d_q\dot{x} \le \int^*_{G/H} \left(\int^*_H \frac{|g_n(xh) - g(xh)|^{p'}}{q(xh)} dh\right) d_q\dot{x}$$

$$\le \int^*_G |g_n(x) - g(x)|^{p'} dx,$$

and similarly

$$m^*_{G/H}(1_X t_n^p)^{1/p} \leq N_p(f - f_n), \quad m^*_{G/H}(1_X u^{p'})^{1/p'} \leq N_{p'}(g),$$

whence

$$m^*_{G/H}(|\lambda - \lambda_n|) \leq \|\!|T|\!\|_p \frac{1}{n}\Big(1 + N_p(f) + N_{p'}(g)\Big).$$

This implies $\lambda \in \mathcal{L}^1(G/H)$ and, for $n \in \mathbb{N}$

$$\Big|\big\langle i(T)[f],[g]\big\rangle - m_{G/H}(\lambda)\Big| \leq \|\!|T|\!\|_p \frac{2}{n}\Big(1 + N_p(f) + N_{p'}(g)\Big).$$

Corollary 6. *Le G be a locally compact group, H a closed subgroup of G, β a Bruhat function for H, $1 < p < \infty$, $f \in \mathcal{L}^p(G)$, $g \in \mathcal{L}^{p'}(G)$, A associated to $|f|^p$, B associated to $|g|^{p'}$ and $T \in CV_p(G)$. For $x \in G$ we put $\lambda(\dot{x}) = 0$ if $\dot{x} \in A \cup B$ and*

$$\lambda(\dot{x}) = \left\langle T\left[\left(\frac{f}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{g}{q^{1/p'}}\right)_{x,H}\right]\right\rangle$$

if $\dot{x} \notin A \cup B$. Then:

1. $(\lambda \circ \omega)\beta q \in \mathcal{L}^1(G)$,
2. $m_G\Big((\lambda \circ \omega)\beta q\Big) = \big\langle i(T)[f],[g]\big\rangle$.

7.3 A Canonical Isometry of $CV_p(H)$ into $CV_p(G)$

Lemma 1. *Let G be a locally compact group, H a closed subgroup, U an open neighborhood of e in G and W an open neighborhood of e in H. Then there is $k \in C_{00}(G)$ with:*

1. $k(x) \geq 0$ *for every* $x \in G$,
2. $\displaystyle\int_H k(h)dh = 1$,
3. $\displaystyle\int_H k(xh)dh \leq 1$ *for every* $x \in G$,
4. $\operatorname{supp} k \subset U$,
5. $(\operatorname{supp} k) \cap H \subset W$.

Proof. There is U_1 open subset of G with $W = H \cap U_1$, K compact neighborhood of e in G with $K \subset U \cap U_1$ and U_2 an open relatively compact subset of G with $K \subset U_2 \subset \overline{U_2} \subset U \cap U_1$. Let ψ be a continuous map of G/H into $[0,1]$ with $\psi(\dot{e}) = 1$

and $\operatorname{supp}\psi \subset \omega(K)$ and φ a continuous map of G into $[0,1]$ with $\varphi(k) = 1$ for every $k \in K$ and $\operatorname{supp}\varphi \subset U_2$. Then

$$\left\{ x \in G \,\middle|\, \int_H \varphi(xh)dh \neq 0 \right\} = AH$$

with $A = \varphi^{-1}(\mathbb{R} \setminus \{0\})$. Let $x \in G$. If $x \in AH$ we set

$$k(x) = \frac{\varphi(x)\psi(\omega(x))}{\int_H \varphi(xh)dh}$$

and for $x \notin AH$ we set $k(x) = 0$. Then k satisfied all the required properties.

We are now able to complete Theorem 13 of Sect. 7.1.

Theorem 2. *Let G be a locally compact group, H a closed subgroup of G and $p > 1$. Then i is a linear isometry of the Banach space $CV_p(H)$ into $CV_p(G)$.*

Proof. Let $T \in CV_p(H)$ and $\varphi, \psi \in C_{00}(H)$. It suffices to prove that

$$\left|\langle T[\varphi], [\psi]\rangle\right| \le \left\|\!\left| i(T) \right|\!\right\|_p N_p(\varphi)N_{p'}(\psi).$$

Let $\varepsilon > 0$. There is W, an open neighborhood of e in H, such that for every $h \in W$ we have

$$N_p\big(\varphi -_{h^{-1}} \varphi\big) < \frac{\delta}{2} \quad \text{and} \quad N_{p'}\big(\psi -_{h^{-1}} \psi\big) < \frac{\delta}{2}$$

where

$$0 < \delta < \min\left\{1, \frac{\varepsilon}{\left(1 + \left\|\!\left|T\right|\!\right\|_p\right)\Big(1 + N_p(\varphi) + N_{p'}(\psi)\Big)}\right\}.$$

By Lemma 1 there is $k \in C_{00}(G)$ with $k(x) \geq 0$ $\int_H k(xh)dh \le 1$ for every $x \in G$, $\int_H k(h)dh = 1$ and $\operatorname{supp} k \cap H \subset W$. For every $x \in G$ we put

$$v(x) = q(x)^{1/p} \int_H k(xh)\varphi(h^{-1})dh \quad \text{and} \quad v(x) = q(x)^{1/p'} \int_H k(xh)\psi(h^{-1})dh.$$

Then $v, w \in C_{00}(G)$.

1. We have

$$N_p\left(Res_H\left(\frac{v}{q^{1/p}}\right) - \varphi\right) \le \frac{\delta}{2} \quad \text{and} \quad N_{p'}\left(Res_H\left(\frac{w}{q^{1/p'}}\right) - \psi\right) \le \frac{\delta}{2}.$$

Let $g \in C_{00}(H)$ with $N_{p'}(g) \le 1$ we have

$$\left|\int_H g(h)\left(\frac{v(h)}{q(h)^{1/p}} - \varphi(h)\right)dh\right| \le \frac{\delta}{2} \quad \text{and thus} \quad N_p\left(Res_H\left(\frac{v}{q^{1/p}}\right) - \varphi\right) \le \frac{\delta}{2}.$$

Similarly

$$N_{p'}\left(Res_H\left(\frac{w}{q^{1/p'}}\right) - \psi\right) \le \frac{\delta}{2}.$$

2. There is an open neighborhood U of e in G relatively compact such that for every $x \in U$ we have

$$N_p\left(\left(\frac{v}{q^{1/p}}\right)_{x,H} - \varphi\right) < \delta \quad \text{and} \quad N_{p'}\left(\left(\frac{w}{q^{1/p'}}\right)_{x,H} - \psi\right) < \delta.$$

By Lemma 8 of Sect. 7.1 there is a relatively compact open neighborhood U of e in G such that for every $x \in U$ we have

$$\left(\int_H \left|\frac{v(xh)}{q(xh)^{1/p}} - \frac{v(h)}{q(h)^{1/p}}\right|^p dh\right)^{1/p} < \frac{\delta}{2} \quad \text{and} \quad \left(\int_H \left|\frac{w(xh)}{q(xh)^{1/p'}} - \frac{w(h)}{q(h)^{1/p'}}\right|^{p'} dh\right)^{1/p'} < \frac{\delta}{2}.$$

3. Let

$$\sigma = \frac{1_{UH}}{m_{G/H}(\omega(U))^{1/p}} \quad \text{and} \quad \tau = \frac{1_{UH}}{m_{G/H}(\omega(U))^{1/p'}}.$$

Then $\frac{|\sigma v|^p}{q}$ and $\frac{|\tau v|^{p'}}{q}$ belong to $\mathcal{T}^+(G)$. For $x \in G$ we have $\left(\frac{|\sigma v|^p}{q}\right)_{x,H} \in \mathcal{T}^+(H)$ (see the proof of Proposition 1 of Sect. 7.2) and

$$m_H^*\left(\left(\frac{|\sigma v|^p}{q}\right)_{x,H}\right) \le \frac{1}{m_{G/H}(\omega(U))} \int_H^* \frac{|v(xh)|^p}{q(xh)} dh.$$

But $\left(\frac{|v|^p}{q}\right)_{x,H} \in C_{00}(H)$. Consequently $\left(\frac{|\sigma v|^p}{q}\right)_{x,H} \in \mathcal{L}^1(H)$. Similarly $\left(\frac{|\tau w|^p}{q}\right)_{x,H} \in \mathcal{L}^1(H)$. Then by Theorem 5 of Sect. 7.2

$$\langle i(T)[\sigma v], [\tau w]\rangle = \int_{G/H} \lambda(\dot{x}) d_q \dot{x}$$

where for every $x \in G$

$$\lambda(\dot{x}) = \left\langle T\left[\left(\frac{\sigma v}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\tau w}{q^{1/p'}}\right)_{x,H}\right]\right\rangle.$$

But for $x \in G$

$$\left(\frac{\sigma v}{q^{1/p}}\right)_{x,H} = \frac{1_{\omega(U)}(\dot{x})}{m_{G/H}(\omega(U))^{1/p}} \left(\frac{v}{q^{1/p}}\right)_{x,H} \quad \text{and} \quad \left(\frac{\tau w}{q^{1/p'}}\right)_{x,H} = \frac{1_{\omega(U)}(\dot{x})}{m_{G/H}(\omega(U))^{1/p'}} \left(\frac{v}{q^{1/p'}}\right)_{x,H}$$

and therefore

$$\lambda(\dot{x}) = \frac{1_{\omega(U)}(\dot{x})}{m_{G/H}(\omega(U))} \left\langle T\left[\left(\frac{v}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{w}{q^{1/p'}}\right)_{x,H}\right]\right\rangle.$$

Consequently

$$m_{G/H}(\lambda) - \langle T[\varphi], [\psi]\rangle = \frac{1}{m_{G/H}(\omega(U))} \int_G 1_{UH}(x)\beta(x)q(x) \times \left\{\left\langle T\left[\left(\frac{v}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{w}{q^{1/p'}}\right)_{x,H}\right]\right\rangle - \langle T[\varphi], [\psi]\rangle\right\} dx.$$

Hence

$$\left|\langle i(T)[\sigma v], [\tau w]\rangle - \langle T[\varphi], [\psi]\rangle\right| \le \frac{1}{m_{G/H}(\omega(U))} \int_G 1_{UH}(x)\beta(x)q(x) \times \left|\left\langle T\left[\left(\frac{v}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{w}{q^{1/p'}}\right)_{x,H}\right]\right\rangle - \langle T[\varphi], [\psi]\rangle\right| dx.$$

But for every $u \in U$ and every $h \in H$ we have

$$\left\langle T\left[\left(\frac{v}{q^{1/p}}\right)_{uh,H}\right], \left[\left(\frac{w}{q^{1/p'}}\right)_{uh,H}\right]\right\rangle = \left\langle T\left[\left(\frac{v}{q^{1/p}}\right)_{u,H}\right], \left[\left(\frac{w}{q^{1/p'}}\right)_{u,H}\right]\right\rangle,$$

this implies

$$\left|\left\langle T\left[\left(\frac{v}{q^{1/p}}\right)_{uh,H}\right], \left[\left(\frac{w}{q^{1/p'}}\right)_{uh,H}\right]\right\rangle - \langle T[\varphi], [\psi]\rangle\right| \le |||T|||_p\, \delta\Big(\delta + N_p(\varphi) + N_{p'}(\psi)\Big).$$

Consequently

$$\Big|\langle i(T)[\sigma v], [\tau w]\rangle - \langle T[\varphi], [\psi]\rangle\Big|$$
$$\le \frac{1}{m_{G/H}(\omega(U))} \int_G 1_{UH}(x)\beta(x)q(x)\, |||T|||_p\, \delta\Big(\delta + N_p(\varphi) + N_{p'}(\psi)\Big) dx < \varepsilon.$$

This implies

$$\Big|\langle T[\varphi], [\psi]\rangle\Big| < \varepsilon + |||i(T)|||_p\, N_p(\sigma v) N_{p'}(\tau w).$$

For every $x \in G$

$$\int_H \frac{|v(xh)|^p}{q(xh)} dh \le N_p(\varphi)^p$$

and therefore $N_p(\sigma v) \le N_p(\varphi)$ and similarly $N_p(\tau w) \le N_{p'}(\psi)$. This finally implies

$$\Big|\langle T[\varphi], [\psi]\rangle\Big| < \varepsilon + |||i(T)|||_p\, N_p(\varphi) N_{p'}(\psi).$$

Remarks. 1. See Derighetti ([28], Théorème 1, p. 72 and Théorème 2, p. 76). See also Lohoué ([85], Théorème 5, p. 190). For p-pseudomeasures the result is due to Herz ([61], Theorem A, p. 91).

2. This proof (and also Definition 7 of Sect. 7.1) was inspired by a work of Gilbert concerning locally compact abelian groups ([50], Proposition 2, p. 141).

Corollary 3. *Let G be a locally compact group, H a closed subgroup of G, and $1 < p < \infty$. Then for every $\mu \in M^1(G)$ we have $\left|\left|\left|\lambda_G^p(1_H\mu)\right|\right|\right|_p = \left|\left|\left|\lambda_H^p(Res_H\mu)\right|\right|\right|_p$.*

Proof. We have $i(Res_H\mu) = 1_H\mu$.

The following lemma is a generalization of Theorem 1 of Sect. 1.2.

Lemma 4. *Let G be a locally compact group, H a closed subgroup, $1 < p < \infty$, $k \in C_{00}(G)$ and $f \in \mathcal{L}^p(H)$. Then:*

1. *$k *_H f \in C(G)$,*
2. *For every $x \in G$ we have $(k *_H f)_{x,H} \in \mathcal{L}^p(H)$,*
3. *$q^{1/p}(k *_H f) \in \mathcal{L}^p(G)$ and $N_p\Big(q^{1/p}(k *_H f)\Big) \le N_p(f)\, N_p(T_H|k|)$.*

Proposition 5. *Let G be a locally compact group, H a closed subgroup of G, and $1 < p < \infty$. Then i is a Banach algebra isomorphism of $CV_p(H)$ into $CV_p(G)$.*

Proof. We have only to prove that $i(ST) = i(S)i(T)$ for $S, T \in CV_p(H)$.

1. First we show that for $S \in CV_p(H)$, $k \in C_{00}(G)$, and $f, g \in \mathcal{L}^p(H)$ with $S[f] = [g]$ we have $i(S)\big[q^{1/p}(k *_H f)\big] = \big[q^{1/p}(k *_H g)\big]$.
   It suffices to prove that for every $\psi \in C_{00}(G)$ we have

$$\Big\langle i(S)\big[q^{1/p}(k *_H f)\big], [\psi]\Big\rangle = \Big\langle \big[q^{1/p}(k *_H g)\big], [\psi]\Big\rangle.$$

   Let A be a subset of G/H associated to $q|k *_H f|^p$. We put

$$\lambda_1(\dot{x}) = \left\langle S\big[(k *_H f)_{x,H}\big], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle$$

   for $\dot{x} \notin A$ and $\lambda_1(\dot{x}) = 0$ if $\dot{x} \in A$. Then

$$\int_{G/H} \lambda_1(\dot{x}) d_q\dot{x} = \Big\langle i(S)\big[q^{1/p}(k *_H f)\big], [\psi]\Big\rangle.$$

   Similarly let B be a subset of G/H associated to $q|k *_H g|^p$ and

$$\lambda_2(\dot{x}) = \left\langle \big[(k *_H g)_{x,H}\big], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle$$

   if $\dot{x} \notin B$ and $\lambda_2(\dot{x}) = 0$ if $\dot{x} \in B$, then

$$\int_{G/H} \lambda_2(\dot{x}) d_q\dot{x} = \Big\langle \big[q^{1/p}(k *_H g)\big], [\psi]\Big\rangle.$$

   But for $\dot{x} \notin A \cup B$ we have $\lambda_1(\dot{x}) = \lambda_2(\dot{x})$ and therefore

$$\Big\langle i(S)\big[q^{1/p}(k *_H f)\big], [\psi]\Big\rangle = \Big\langle \big[q^{1/p}(k *_H g)\big], [\psi]\Big\rangle.$$

2. Next we prove that for $S, T \in CV_p(H)$, $k \in C_{00}(G)$ and $f \in \mathcal{L}^p(H)$ one has

$$i(ST)\Big[q^{\frac{1}{p}}(k *_H f)\Big] = i(S)i(T)\Big[q^{\frac{1}{p}}(k *_H f)\Big].$$

   Let $l, m \in \mathcal{L}^p(H)$ with $[m] = ST[f]$ and $[l] = T[f]$. We get

$$i(ST)\Big[q^{1/p}(k *_H f)\Big] = \Big[q^{1/p}(k *_H m)\Big].$$

But

$$i(S)i(T)\Big[q^{1/p}(k *_H f)\Big] = i(S)\Big[q^{1/p}(k *_H l)\Big] = \Big[q^{1/p}(k *_H m)\Big].$$

3. For $\varphi \in \mathcal{L}^p(G)$ and $\varepsilon > 0$ there is $k \in C_{00}(G)$ and $f \in C_{00}(H)$ with

$$N_p\Big(\varphi - q^{1/p}(k *_H f)\Big) < \varepsilon.$$

We choose $\varphi' \in C_{00}(C)$ with

$$N_p(\varphi - \varphi') < \frac{\varepsilon}{2}$$

and U a compact neighborhood of e in H. There is V an open neighborhood of e in H with $V \subset U$ and

$$N_p\big(k - k_{h^{-1}}\Delta_H(h^{-1})\big) < \frac{\varepsilon}{2\Big(1 + \sup\Big\{q(x)^{1/p}\Big| x \in (\operatorname{supp}\varphi')U\Big\}\Big)}$$

for every $h \in V$, where $k = \dfrac{\varphi'}{q^{1/p}}$. It suffices then to choose $f \in C_{00}(H)$ with $f \geq 0$, $\operatorname{supp} f \subset V$ and $\displaystyle\int_H f(h)dh = 1$.

4. $i(ST)[\varphi] = i(S)i(T)[\varphi]$ for $S, T \in CV_p(H)$ and $\varphi \in \mathcal{L}^p(G)$.
Let $\varepsilon > 0$. There is $k \in C_{00}(G)$ and $f \in C_{00}(H)$ with

$$N_p\Big(\varphi - q^{1/p}(k *_H f)\Big) < \frac{\varepsilon}{2(1 + |||S|||_p\, |||T|||_p)}.$$

Then

$$\Big\|\Big(i(ST)-i(S)i(T)\Big)[\varphi]\Big\|_p \leq \Big\|\Big(i(ST)-i(S)i(T)\Big)\Big([\varphi]-\big[q^{1/p}(k*_H f)\big]\Big)\Big\|_p < \varepsilon.$$

Remark. See Théorème 5, p. 54 of [31].

7.4 The Support of $i(T)$

Lemma 1. *Let G be a locally compact group, H a closed subgroup, $1 < p < \infty$ and i the map of $CV_p(H)$ into $CV_p(G)$ defined at Sect. 7.1. Let $\varepsilon > 0$ $\varphi, \psi \in C_{00}(H)$ and U_0 a neighborhood of e in G. Then there is U_1 open neighborhood of e in G and $k \in C_{00}(G)$ such that:*

1. $U_1 \subset U_0$,
2. $k(x) \geq 0$ *for every* $x \in G$,
3. $\operatorname{supp} k \subset U_0$,
4. $\int_H k(xh)dh \leq 1$ *for every* $x \in G$,
5. $\int_H k(h)dh = 1$,
6. *For every open neighborhood* V *of* e *in* G *with* $V \subset U_1$ *and for every* $T \in CV_p(H)$

   $(6)_1$
$$\left|\left\langle i(T)\left[\frac{1_{VH}\, q^{1/p}\,(k *_H \varphi)}{m_{G/H}(\omega(V))^{1/p}}\right], \left[\frac{1_{VH}\, q^{1/p'}\,(k *_H \psi)}{m_{G/H}(\omega(V))^{1/p'}}\right]\right\rangle - \langle T[\varphi], [\psi]\rangle\right| \leq \varepsilon \, |||T|||_p \,,$$

   $(6)_2$
$$N_p\left(\frac{1_{VH}\, q^{1/p}(k *_H \varphi)}{m_{G/H}(\omega(V))^{1/p}}\right) \leq N_p(\varphi),$$

   $(6)_3$
$$N_{p'}\left(\frac{1_{VH}\, q^{1/p'}\,(k *_H \psi)}{m_{G/H}(\omega(V))^{1/p'}}\right) \leq N_{p'}(\psi).$$

Proof. Let
$$0 < \delta < \min\left\{1, \frac{\varepsilon}{1 + N_p(\varphi) + N_{p'}(\psi)}\right\}.$$

There is W, open neighborhood of e in H, such that $N_p(\varphi -_{h^{-1}} \varphi) < \frac{\delta}{2}$ and $N_{p'}(\psi -_{h^{-1}} \psi) < \frac{\delta}{2}$ for every $h \in W$. According to Lemma 1 of Sect. 7.3 there is $k \in C_{00}(G)$ with $k(x) \geq 0$, $\int_H k(xh)dh \leq 1$ for every $x \in G$, $\int_H k(h)dh = 1$, $\operatorname{supp} k \subset U_0$, and $(\operatorname{supp} k) \cap H \subset W$. Let $v = q^{\frac{1}{p}}(k *_H \varphi)$ and $w = q^{\frac{1}{p'}}(k *_H \psi)$. As in the step (1) of the proof of Theorem 2 of Sect. 7.3 we have

$$N_p\left(Res_H\left(\frac{v}{q^{1/p}}\right) - \varphi\right) \leq \frac{\delta}{2} \quad \text{and} \quad N_{p'}\left(Res_H\left(\frac{w}{q^{\frac{1}{p'}}}\right) - \psi\right) \leq \frac{\delta}{2}.$$

By Lemma 8 of Sect. 7.1 there is U_1, open neighborhood of e in G, relatively compact with $U_1 \subset U_0$

$$\left(\int_H \left|\frac{v(xh)}{q(xh)^{1/p}} - \frac{v(h)}{q(h)^{1/p}}\right|^p dh\right)^{1/p} < \frac{\delta}{2} \quad \text{and} \quad \left(\int_H \left|\frac{w(xh)}{q(xh)^{1/p'}} - \frac{w(h)}{q(h)^{1/p'}}\right|^{p'} dh\right)^{1/p'} < \frac{\delta}{2}$$

for every $x \in U_1$.

Let V be an open neighborhood of e in G with $V \subset U_1$. For

$$\sigma = \frac{1_{VH}}{m_{G/H}(\omega(V))^{1/p}} \quad \text{and} \quad \tau = \frac{1_{VH}}{m_{G/H}(\omega(V))^{1/p'}}$$

we have

$$\left|\langle i(T)[\sigma v], [\tau w]\rangle - \langle T[\varphi], [\psi]\rangle\right| \leq \varepsilon \, |||T|||_p$$

for every $T \in CV_p(H)$.

Lemma 2. *Let G be a locally compact group, $1 < p < \infty$, $T \in CV_p(G)$, $x_0 \in G$, U, V open subsets of G with $e \in U$, $x_0 \in V$ and $\langle T[\varphi], [\psi]\rangle = 0$ for every $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset U$ and $\operatorname{supp}\psi \subset V$. Then for every $\varphi, \psi \in C_{00}(G; \mathbb{C})$ with $\operatorname{supp}\varphi \subset U$, $\operatorname{supp}\psi \subset V$, every $r \in \mathcal{L}^p(G)$ and every $s \in \mathcal{L}^{p'}(G)$ we have $\langle T[\varphi\, r], [\psi\, s]\rangle = 0$.*

Proof. There are (r_n) and (s_n) sequences of $C_{00}(G)$ with $\lim N_p(r_n - r) = 0$ and $\lim N_{p'}(s_n - s) = 0$. There is $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ $N_p(r_n - r) < 1$. Then for every $n \geqslant n_0$ we have

$$\left|\langle T[\varphi\, r], [\psi\, s]\rangle\right| \leq |||T|||_p \, \|\varphi\|_u \|\psi\|_u N_p(r_n - r) N_{p'}(s) + |||T|||_p \, \|\varphi\|_u \|\psi\|_u N_{p'}(s_n - s)\big(N_p(r) + 1\big).$$

Theorem 3. *Let G be a locally compact group, H a closed subgroup and $p \in (1, \infty)$. Then for every $T \in CV_p(H)$ we have $\operatorname{supp} i(T) = \operatorname{supp} T$.*

Proof. 1. $\operatorname{supp} i(T) \subset H$.

Let $x_0 \in G \setminus H$. There is U and V open subsets of G/H with $U \cap V = \emptyset$, $\omega(e) \in U$, and $\omega(x_0) \in V$. Let $U_1 = \omega^{-1}(U)$ and $V_1 = \omega^{-1}(V)$. Then $U_1 \cap V_1 = \emptyset$. Let $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset U_1$ and $\operatorname{supp}\psi \subset V_1$. For $x \in G \setminus V_1$ we have

$$\left(\frac{\psi}{q^{1/p}}\right)_{x,H} = 0$$

and therefore

$$\langle i(T)[\varphi\,], [\psi\,]\rangle = \int_{V_1} \beta(x) q(x) \left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle dx.$$

But if $x \in V_1$ then $x \notin U_1$ and therefore

$$\left(\frac{\varphi}{q^{1/p}}\right)_{x,H} = 0$$

consequently $\langle i(T)[\varphi], [\psi]\rangle = 0$. This implies that $x_0 \notin \operatorname{supp} i(T)$.
2. $\operatorname{supp} i(T) \subset \operatorname{supp} T$.

Let $h_0 \in H$ with $h_0 \notin \operatorname{supp} T$. There is V_0, V_1 open subsets of H with $e \in V_0$, $h_0 \in V_1$ and $\langle T[\varphi], [\psi]\rangle = 0$ for every $\varphi, \psi \in C_{00}(H)$ with $\operatorname{supp}\varphi \subset V_0$ and $\operatorname{supp}\psi \subset V_1$. There is U_0, U_1 open subsets of G with $U_0 \cap H = V_0$ and $U_1 \cap H = V_1$. There is U_2 open neighborhood of e in G such that $U_2 = U_2^{-1}$, $U_2^2 \subset U_0$ and $U_2^2 h_0 \subset U_1$. Choose finally U_3, open neighborhood of e in G, with $\overline{U}_3 \subset U_2$ and $\overline{U}_3$ compact. Let $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset U_3$ and $\operatorname{supp}\psi \subset U_3 h_0$. For $x \in G \setminus U_3 H$ we have

$$\left(\frac{\varphi}{q^{1/p}}\right)_{x,H} = 0$$

and therefore

$$\langle i(T)[\varphi], [\psi]\rangle = \int_{U_3 H} \beta(x) q(x) \left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle dx.$$

Let $x \in U_3 H$. We have $x = uh$ with $u \in U_3$ and $h \in H$. Then

$$\left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle = \left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{u,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{u,H}\right]\right\rangle.$$

From

$$\operatorname{supp}\left(\frac{\varphi}{q^{1/p}}\right)_{u,H} \subset V_0 \quad \text{and} \quad \operatorname{supp}\left(\frac{\psi}{q^{1/p'}}\right)_{u,H} \subset V_1$$

we obtain

$$\left\langle T\left[\left(\frac{\varphi}{q^{1/p}}\right)_{u,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{u,H}\right]\right\rangle = 0 \quad \text{and therefore} \quad \langle i(T)[\varphi], [\psi]\rangle = 0,$$

consequently $h_0 \notin \operatorname{supp} i(T)$.
3. $\operatorname{supp} T \subset i(T)$.

Let $h_0 \in H$ with $h_0 \notin \operatorname{supp} i(T)$. There is U_0, V_0 open subsets of G with $e \in U_0$, $h_0 \in V_0$ and $\langle i(T)[a], [b]\rangle = 0$ for every $a, b \in C_{00}(G)$ with $\operatorname{supp} a \subset U_0$ and $\operatorname{supp} b \subset V_0$. There is U_1 open neighborhood of e in G, V_2 open neighborhood of h_0 in G such that $U_1^2 \subset U_0$ and $U_1 V_2 \subset V_0$. Let $W_1 = U_1 \cap H$ and $W_2 = V_2 \cap H$.

Consider $\varphi, \psi \in C_{00}(H)$ with $\operatorname{supp}\varphi \subset W_1$ and $\operatorname{supp}\psi \subset W_2$. We show that $\langle T[\varphi], [\psi]\rangle = 0$.

Let $\varepsilon > 0$.

According to Lemma 1 there is V_3 open neighborhood of e in G and $k \in C_{00}(G)$ such that:

1. $V_3 \subset U_1$,
2. $k(x) \geq 0$ for every $x \in G$,
3. $\operatorname{supp} k \subset U_1$,
4. $\int_H k(xh)dh \leq 1$ for every $x \in G$,
5. $\int_H k(h)dh = 1$,
6. $$\left|\left\langle i(T)\left[\frac{1_{V_3H}\, q^{1/p}\, (k *_H \varphi)}{m_{G/H}(\omega(V_3))^{1/p}}\right], \left[\frac{1_{V_3H}\, q^{1/p'}\, (k *_H \psi)}{m_{G/H}(\omega(V_3))^{1/p'}}\right]\right\rangle - \langle T[\varphi], [\psi]\rangle\right| \leq \varepsilon.$$

But $\operatorname{supp} k *_H \varphi \subset U_0$ and $\operatorname{supp} k *_H \psi \subset V_0$. Let

$$r = \frac{q^{1/p} 1_{V_3 H \cap \operatorname{supp} k *_H \varphi}}{m_{G/H}(\omega(V_3))^{1/p}} \quad \text{and} \quad s = \frac{q^{1/p'} 1_{V_3 H \cap \operatorname{supp} k *_H \psi}}{m_{G/H}(\omega(V_3))^{1/p'}}.$$

Lemma 2 implies

$$\left\langle i(T)\big[r(k *_H \varphi)\big], \big[s(k *_H \psi)\big]\right\rangle = 0$$

and therefore

$$\left|\langle T[\varphi], [\psi]\rangle\right| < \varepsilon. \qquad \text{Consequently} \quad \langle T[\varphi], [\psi]\rangle = 0.$$

We obtain that $h_0 \notin \operatorname{supp} T$.

Remark. This result is due to Anker ([2], p. 631 and [1], Lemme IV.6, p.109).

7.5 Theorems of de Leeuw and Saeki

In this paragraph, G is a locally compact abelian group, H an arbitrary closed subgroup, m_G a Haar measure on G, m_H a Haar measure on H, ω the canonical map of G onto G/H and $m_{G/H} = \dfrac{m_G}{m_H}$. For $\chi \in H^\perp$ we set $\rho(\chi)(\omega(x)) = \chi(x)$ where $x \in G$. The dual map $\widehat{\omega}$ is a topological isomorphism of $\widehat{G/H}$ onto $H^\perp$. Let γ be the canonical map of $\widehat{G}$ onto $\widehat{G}/H^\perp$. Then the dual map $\widehat{\gamma}$ is a topological isomorphism of $\widehat{\widehat{G}/H^\perp}$ onto $H^{\perp\perp}$. For $h \in H$ we set $(\sigma(h)(\gamma(\chi)) = \chi(h)$ for every $\chi \in \widehat{G}$. Finally for $\chi \in \widehat{G}$ we also set $\tau(\gamma(\chi)) = Res_H \chi$. By Hewitt and

Ross ([67], p. 242–246) we have

$$\tau^{-1}(m_{\hat{H}}) = \frac{m_{\hat{G}}}{m_{H^\perp}} \quad \text{where} \quad m_{H^\perp} = \widehat{\omega}\left(m_{\widehat{G/H}}\right).$$

Observe that for $\mu \in M^1(H)$ we have $\widehat{i(\mu)} = \widehat{\mu}\circ\tau\circ\gamma$. We show that this relation is verified for every $T \in CV_p(H)$.

Theorem 1. *Let G be a locally compact abelian group, $1 < p < \infty$, H a closed subgroup of G, $T \in CV_p(H)$, $f \in \mathbb{C}^{\hat{G}}$, $g \in \mathbb{C}^{\hat{H}}$ with $\dot{f} = \widehat{i(T)}$ and $\dot{g} = \widehat{T}$. Then $f(\chi) = g(\tau(\gamma(\chi)))$ $m_{\hat{G}}$-locally almost everywhere on $\widehat{G}$.*

Proof. Let $\varphi, \psi \in C_{00}(G)$. We have

$$\left\langle i(T)[\varphi], [\psi]\right\rangle = \left\langle (\widehat{i(T)})\mathcal{F}[\varphi], \mathcal{F}[\psi]\right\rangle = \int_{\hat{G}} f(\chi)\widehat{\varphi}(\chi)\overline{\widehat{\psi}(\chi)}dm_{\hat{G}}(\gamma).$$

On the other hand

$$\left\langle i(T)[\varphi], [\psi]\right\rangle = \int_{G/H} \left\langle T[\varphi_{x,H}], [\psi_{x,H}]\right\rangle dm_{G/H}(\omega(x)).$$

For every $x \in G$ we have

$$\left\langle T[\varphi_{x,H}], [\psi_{x,H}]\right\rangle = \int_{\hat{H}} g(\gamma)\widehat{\varphi_{x,H}}(\chi)\overline{\widehat{\psi_{x,H}}(\chi)}dm_{\hat{H}}(\gamma),$$

consequently

$$\left\langle i(T)[\varphi], [\psi]\right\rangle = \int_{\hat{H}} \left(\int_{G/H} g(\gamma)\widehat{\varphi_{x,H}}(\chi)\overline{\widehat{\psi_{x,H}}(\chi)}dm_{G/H}(\omega(x)) \right) dm_{\hat{H}}(\gamma).$$

But for every $x \in G$ and every $\chi \in \widehat{H}$ we have

$$\widehat{\varphi_{x,H}}(\chi)\overline{\widehat{\psi_{x,H}}(\chi)} = \int_H \chi(h') \left(\int_H \varphi_{x,H}(h)\overline{\psi_{x,H}(hh')}dm_H(h) \right) dm_H(h'),$$

and thus

$$\begin{aligned}&\left\langle i(T)[\varphi], [\psi]\right\rangle \\ &= \int_{\hat{H}} g(\chi) \int_{G/H} \int_H \chi(h') \left(\int_H \varphi_{x,H}(h)\overline{\psi_{x,H}(hh')}dm_H(h) \right) dm_H(h')dm_{G/H}(\omega(x))dm_{\hat{H}}(\chi)\end{aligned}$$

$$= \int_{\hat{H}} g(\chi) \int_{H} \chi(h') \int_{G/H} \left(\int_{H} \varphi_{x,H}(h) \overline{\psi_{x,H}(hh')} dm_H(h) \right) dm_{G/H}(\omega(x)) dm_H(h') dm_{\hat{H}}(\chi)$$

$$= \int_{\hat{H}} g(\chi) \int_{H} \chi(h') \left(\int_{G} \varphi(x) \overline{\psi(xh')} dm_G(x) \right) dm_H(h') dm_{\hat{H}}(\chi)$$

$$= \int_{\hat{H}} g(\chi) \left(\int_{H} \overline{\chi(h)} (\varphi * \psi^{\sim})(h) dm_H(h) \right) dm_{\hat{H}}(\chi) = \int_{\hat{H}} g(\chi) \big(Res_H(\varphi * \psi^{\sim})\big)^{\widehat{}}(\chi) dm_{\hat{H}}(\chi)$$

$$= \int_{\hat{G}/H^{\perp}} g(\tau(\gamma(\chi))) \big(Res_H(\varphi * \psi^{\sim})\big)^{\widehat{}}(\tau(\gamma(\chi))) dm_{\hat{G}/H^{\perp}}(\gamma(\chi)).$$

Taking into account that $\big(Res_H(\varphi * \psi^{\sim})\big)^{\widehat{}} \circ \tau = T_{H^{\perp}}(\varphi * \psi^{\sim})^{\widehat{}}$ we get

$$\langle i(T)[\varphi], [\psi] \rangle = \int_{\hat{G}/H^{\perp}} g(\tau(\gamma(\chi))) \Big(T_{H^{\perp}}(\varphi * \psi^{\sim})^{\widehat{}} \Big) (\gamma(\chi)) dm_{\hat{G}/H^{\perp}}(\gamma(\chi))$$

$$= \int_{\hat{G}} g(\tau(\gamma(\chi))) \,\widehat{\varphi}\,(\chi) \overline{\widehat{\psi}(\chi)} dm_{\hat{G}}(\chi),$$

and therefore $f(\chi) = g(\tau(\gamma(\chi)))$ $m_{\hat{G}}$-locally almost everywhere on $\widehat{G}$.

Remarks. 1. For $G = \mathbb{R}$ and $H = \mathbb{Z}$ this result (together with the isometry of Theorem 2 of Sect. 7.3) is due to de Leeuw ([74], Theorem 4.5, p. 377).
2. Theorem 1 is due to Saeki ([108], Corollary 3.5, p. 417).
3. For the above proof cf [31] (Proposition 11, p. 60).

Corollary 2. *Let G be a locally compact abelian group, H a closed subgroup of G and $g \in \mathcal{L}^{\infty}(\widehat{H})$. Then $sp(g \circ \tau \circ \gamma)^{\cdot} = sp\,\dot{g}$ and $\|(g \circ \tau \circ \gamma)^{\cdot}\|_{\infty} = \|\dot{g}\|_{\infty}$.*

Proof. Let $T = \Lambda_{\hat{G}}(\dot{g})$ (see Sect. 1.3, Definition 2). We have $T \in CV_2(H)$, $\widehat{T} = \dot{g}$ and $sp\,\widehat{T} = (\operatorname{supp} T)^{-1}$ by Theorem 3 of Sect. 6.1. By Theorem 1 $g \circ \tau \circ \gamma \in i(T)^{\widehat{}}$. The results then follow from $\operatorname{supp} i(T) = \operatorname{supp} T$ (Theorem 3 of Sect. 7.4), Theorem 1 of Sect. 1.3 and Theorem 2 of Sect. 7.3.

Remarks. 1. We have $sp(g \circ \tau \circ \gamma)^{\cdot} \subset H$.
2. See Reiter and Stegeman [105], Proposition 7.1.20, p. 198.
3. This corollary will be completed in Sect. 7.7.

7.6 The Image of the Map *i*

The essential result of the paragraph and one of the most important in the present book is the Theorem 10 which says that each $T \in CV_p(G)$ supported by H is an $i(S)$.

Proposition 1. *Let G be a locally compact group, H a closed subgroup of G, $1 < p < \infty$ and $\mu \in M^1(G)$ with $\operatorname{supp}(\lambda_G^p(\mu)) \subset H$. Then $\lambda_G^p(\mu) = i(\lambda_H^p(Res_H \mu))$.*

Proof. We have $\mu = 1_H \mu = i(Res_H \mu)$. By the remark after Definition 7 of Sect. 7.1, we get $\lambda_G^p(\mu) = i(\lambda_H^p(Res_H \mu))$.

Remark. For G abelian and $p = 2$ see Reiter and Stegeman ([105], Proposition 5.4.7, p. 162.)

We intend to generalize Proposition 1 to all convolution operators of G.

Definition 1. Let G be a locally compact group, we denote by $\mathcal{A}_{00}(G)$ the linear span of the set $\left\{r * s \middle| r, s \in C_{00}(G)\right\}$.

Lemma 2. *Let G be a locally compact group, $1 < p < \infty$, $T \in CV_p(G)$ and $f \in \mathcal{A}_{00}(G)$. Then there is an unique $g \in C(G) \cap \mathcal{L}^p(G)$ with $T[f] = [g]$.*

Proof. It suffices to verify the existence of g. We have $f = \sum_{j=1}^{n} r_j * s_j$ and therefore $T[f] = \sum_{j=1}^n [r_j] * T[s_j] = \sum_{j=1}^n [r_j * \varphi_j]$ with $\varphi_j \in T[s_j]$. Then $r_j * \varphi_j \in C(G) \cap \mathcal{L}^p(G)$, and so $T[f] = [g]$ with $g = \sum_{j=1}^{n} r_j * \varphi_j$.

Definition 2. The function g of Lemma 2 is denoted by $a_{T,f}$.

Lemma 3. *Let G be a locally compact group and $1 < p < \infty$. Then*

1. *$(T, f) \mapsto a_{T,f}$ is a bilinear map of $CV_p(G) \times \mathcal{A}_{00}(G)$ into $\mathcal{L}^p(G)$;*
2. *$a_{T,{}_x f} = {}_x(a_{T,f})$ for $x \in G$.*

Definition 3. For $f \in \mathcal{L}^\infty(G)$, $1 < p < \infty$ and for every $\varphi \in \mathcal{L}^p(G)$ we put $M_f[\varphi] = [\varphi f]$.

Proposition 4. *Let G be a locally compact group, H a closed subgroup of G, $1 < p < \infty$, $S \in CV_p(H)$ and $f \in \mathcal{L}^\infty(G/H, m_{G/H})$. Then $i(S) \circ M_{f \circ \omega} = M_{f \circ \omega} \circ i(S)$.*

Proof. Let $\varphi, \psi \in C_{00}(G)$. For $x \in G$ we have

$$\left(\frac{f \circ \omega\, \varphi}{q^{1/p}}\right)_{x,H} = f(\omega(x)) \frac{\varphi_{x,H}}{\left(q^{1/p}\right)_{x,H}}.$$

By Theorem 5 of Sect. 7.2

$$\begin{aligned}\left\langle i(S) M_{f \circ \omega}[\varphi], [\psi] \right\rangle &= \int_{G/H} f(\omega(x)) \left\langle S\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle dm_{G/H}(\omega(x)) \\ &= \int_{G/H} \left\langle S\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\overline{f(\omega(x))}\left(\frac{\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle dm_{G/H}(\omega(x))\end{aligned}$$

$$= \int_{G/H} \left\langle S\left[\left(\frac{\varphi}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{\overline{f\circ\omega}\,\psi}{q^{1/p'}}\right)_{x,H}\right]\right\rangle dm_{G/H}(\omega(x))$$

$$= \left\langle i(S)[\varphi], [\overline{f\circ\omega}\,\psi]\right\rangle = \left\langle f\circ\omega i(S)[\varphi], [\psi]\right\rangle = \left\langle M_{f\circ\omega}(i(S))[\varphi], [\psi]\right\rangle.$$

We now need the notion of carrable set. We refer for this notion to Dinculeanu [34], Chap. 5, Sect. 6.1, p. 362.

Lemma 5. *Let G be a locally compact group, H a closed subgroup of G, $1 < p < \infty$ and $T \in CV_p(G)$ with $\operatorname{supp} T \subset H$. For every open relatively compact and $m_{G/H}$-carrable neighborhood V of $\dot{e}$ we then have $T\circ M_{1_{\omega^{-1}(V)}} = M_{1_{\omega^{-1}(V)}}\circ T$.*

Proof. Let $f = 1_{\omega^{-1}(V)}$ and $\varphi \in C_{00}(G)$.
(I) If $\operatorname{supp}\varphi \subset \omega^{-1}(V)$ then $TM_f[\varphi] = M_f T[\varphi]$.

We have $f\varphi = \varphi$. Let W be an open neighborhood of e in G with $(\operatorname{supp}\varphi)W \subset \omega^{-1}(V)$, and $\eta > 0$. There is $\psi \in C_{00}(G;\mathbb{R})$ with $\operatorname{supp}\psi \subset W$ and

$$N_p(\varphi - \varphi * \psi) < \frac{\eta}{2\big(1 + |||T|||_p\big)}.$$

We have $\operatorname{supp}(\varphi * \psi) \subset \omega^{-1}(V)$. From Lemma 1 of Sect. 6.4 we get

$$\operatorname{supp}\Big(T([\varphi * \psi])m_G\Big) \subset \omega^{-1}(V), \quad \text{and consequently}\ \operatorname{supp} a_{T,\varphi*\psi} \subset \omega^{-1}(V).$$

This implies

$$f a_{T,\varphi*\psi} = a_{T,\varphi*\psi} \quad \text{and therefore}\ M_f T[\varphi * \psi] = T[\varphi * \psi].$$

From $f(\varphi * \psi) = \varphi * \psi$ it follows $TM_f[\varphi * \psi] = T[\varphi * \psi]$ and finally

$$M_f T[\varphi * \psi] = TM_f[\varphi * \psi].$$

Then

$$\|TM_f([\varphi]) - M_f T([\varphi])\|_p \le 2\,|||T|||_p\, N_p(\varphi - \varphi * \psi) < \eta.$$

(II) If $(\operatorname{supp}\varphi) \cap \overline{\omega^{-1}(V)} = \emptyset$ then $TM_f[\varphi] = M_f T[\varphi] = 0$.

There is Z_1 open neighborhood of $\operatorname{supp}\varphi$, with $Z_1 \cap \overline{\omega^{-1}(V)} = \emptyset$. There is Z_2 open neighborhood of e in G with $(\operatorname{supp}\varphi)Z_2 \subset Z_1$. Let $\eta > 0$. There is $\psi \in C_{00}(G)$ with $\operatorname{supp}\psi \subset Z_2$ and

$$N_p(\varphi - \varphi * \psi) < \frac{\eta}{(1 + |||T|||_p)}.$$

We have $f a_{T,\varphi*\psi} = 0$. Suppose indeed the existence of $x \in G$ with

$$f(x)a_{T,\varphi*\psi}(x) \neq 0.$$

Then $x \in \omega^{-1}(V)$ and $x \in \operatorname{supp} a_{T,\varphi*\psi}$. We have $x \in \operatorname{supp}\Big((T[\varphi * \psi])m_G\Big)$ then $x = zh$ with $z \in Z_1$ and $h \in H$. This implies $z \in Z_1 \cap \overline{\omega^{-1}(V)}$, a contradiction. We obtain that $M_f T[\varphi * \psi] = 0$. The inequality $\|M_f T[\varphi]\|_p < \eta$ implies $M_f T[\varphi] = 0$. Finally from $f\varphi = 0$ we deduce $M_f[\varphi] = 0$ and $TM_f[\varphi] = 0$.
(III) $TM_f[\varphi] = M_f T[\varphi]$.

Let $K = \operatorname{supp}\varphi$ and $\varepsilon > 0$. We have $m_G(K \cap Fr(\omega^{-1}(V))) = 0$ because is V is carrable (for A subset of a topological space X FrA denotes the border of A). There is therefore U_1 open relatively compact subset of G with $K \cap Fr(\omega^{-1}(V)) \subset U_1$ and

$$m_G(U_1) < \frac{\varepsilon^p}{2^p\big(1 + \|\!|T|\!\|_p\big)^p\big(1 + N_\infty(\varphi)\big)^p}.$$

Choose:

1. U_2 open subset of G such that $\overline{U}_2$ is compact, $K \cap Fr(\omega^{-1}(V)) \subset U_2$ and $\overline{U}_2 \subset U_1$,
2. U_3 open subset of G with $K \setminus (\omega^{-1}(V) \cup U_2) \subset U_3$ and $U_3 \cap \overline{\omega^{-1}(V)} = \emptyset$,
3. U_4 open subset of G with $K \setminus (\omega^{-1}(V) \cup U_2) \subset U_4 \subset \overline{U}_4 \subset U_3$ and $\overline{U}_4$ compact,
4. U_5 open subset of G with $\overline{U}_5$ compact and

$$\Big(K \cap \overline{\omega^{-1}(V)}\Big) \setminus U_2 \subset U_5 \subset \overline{U}_5 \subset \omega^{-1}(V).$$

Then $K \subset U_2 \cup U_4 \cup U_5$. Hence there is $\tau_2, \tau_4, \tau_5 \in C_{00}(G)$ with :

1. $0 \le \tau_j \le 1_G$ for $j \in \{2, 4, 5\}$,
2. $\tau_j(G \setminus U_j) = 0$ for $j \in \{2, 4, 5\}$,
3. $Res_K(\tau_2 + \tau_4 + \tau_5) = 1_K$.

Let $\varphi_2 = \tau_2\varphi$, $\varphi_4 = \tau_4\varphi$ and $\varphi_5 = \tau_5\varphi$. We have $\varphi = \varphi_2+\varphi_4+\varphi_5$, $\operatorname{supp}\varphi_2 \subset U_1$, $\operatorname{supp}\varphi_4 \subset U_3$ and $\operatorname{supp}\varphi_5 \subset \omega^{-1}(V)$. From $U_3 \cap \overline{\omega^{-1}(V)} = \emptyset$ and (II) we get

$$TM_f[\varphi_4] = M_f T[\varphi_4] = 0.$$

From (I) and $\operatorname{supp}\varphi_5 \subset \omega^{-1}(V)$ we have

$$TM_f[\varphi_5] = M_f T[\varphi_5],$$

this implies

$$TM_f[\varphi] - M_f T[\varphi] = TM_f[\varphi_2] - M_f T[\varphi_2]$$

and therefore

$$\|TM_f[\varphi] - M_f T[\varphi]\|_p \le 2\,\|T\|_p\, N_p(\varphi_2) \quad \text{but}$$
$$N_p(\varphi_2)^p \le N_\infty(\varphi)^p m_G(U_1) < \frac{\varepsilon^p}{2^p(1+\|T\|_p)^p}$$

and finally

$$\|TM_f[\varphi] - M_f T[\varphi]\|_p < \varepsilon.$$

The following two assertions are straightforward.

Lemma 6. *Let G be a locally compact group, H a closed subgroup, $\varphi \in C_{00}(H)$ and U an open neighborhood of* $\operatorname{supp}\varphi$ *in G. Then there is $\psi \in C_{00}(G)$ with $Res_H\psi = \varphi$ and* $\operatorname{supp}\psi \subset U$.

Lemma 7. *Let G be a locally compact group, H a closed subgroup, $f \in C_{00}(G)$, $1 < p < \infty$, $\varepsilon > 0$ and U an open neighborhood of e in G. Then there is V open neighborhood of e in G such that:*

i. $V \subset U$,

ii. For every $u \in C_{00}(G)$ with $u \ge 0$, $\operatorname{supp} u \subset V$, *and* $\int_G u(y)dy = 1$ *we have*

$$N_p\Big(Res_H f - Res_H(u * f)\Big) < \varepsilon.$$

Lemma 8. *Let G be a locally compact group, H a closed subgroup, $1 < p < \infty$, $\varepsilon > 0$, $\varphi \in C_{00}(H)$ and let $w \in (0,\infty)^G$ be continuous on G. Then there is $r, s \in C_{00}(G)$ with*

$$N_p\Big(\varphi - Res_H(w(r * s))\Big) < \varepsilon.$$

Proof. Let U be a relatively compact neighborhood of $\operatorname{supp}\varphi$ in G. By Lemma 6 there is $s \in C_{00}(G)$ with $\operatorname{supp} s \subset U$ and

$$Res_H s = \frac{\varphi}{Res_H w}.$$

There is V open neighborhood of e in G with $V\operatorname{supp}\varphi \subset U$. By Lemma 7 there is $r \in C_{00}(G;\mathbb{R})$ with $\operatorname{supp} r \subset V$ and

$$N_p\Big(Res_H s - Res_H(r * s)\Big) \le \frac{\varepsilon}{m}$$

where $m = \max\Big\{w(x)\Big| x \in \overline{U}\Big\}$. But

$$\int_H |\varphi(h) - w(h)(r * s)(h)|^p dh = \int_H w(h)^p \Big|\frac{\varphi(h)}{w(h)} - (r * s)(h)\Big|^p dh.$$

From $\operatorname{supp}(r * s) \subset U$ we get

$$\int_H w(h)^p \left| \frac{\varphi(h)}{w(h)} - (r * s)(h) \right|^p dh \leq m^p \int_H |s(h) - r * s(h)|^p dh$$

and consequently

$$N_p\Big(\varphi - Res_H(w(r * s))\Big) < \varepsilon.$$

Proposition 9. *Let G be a locally compact group, H a closed subgroup and $1 < p < \infty$. Then:*

1. $\Big\{Res_H(r * s)\Big| r, s \in C_{00}(G)\Big\}$ *is dense in* $\mathcal{L}^p(H)$,
2. $\left\{ Res_H\left(\frac{r * s}{q^{1/p}}\right) \middle| r, s \in C_{00}(G) \right\}$ *is dense in* $\mathcal{L}^p(H)$,
3. $\Big\{Res_H\Big(q^{1/p}(r * s)\Big)\Big| r, s \in C_{00}(G)\Big\}$ *is dense in* $\mathcal{L}^{p'}(H)$.

Theorem 10. *Let G be a locally compact group, H a closed subgroup, $1 < p < \infty$ and $T \in CV_p(G)$ with* $\operatorname{supp} T \subset H$. *Then there is $S \in CV_p(H)$ with $i(S) = T$.*

Proof. For $f, g \in \mathcal{A}_{00}(G)$ we have $a_{T,f}\overline{g} \in C_{00}(G)$ and consequently $Res_H (a_{T,f}\overline{g}) \in C_{00}(H)$. For every $f, g \in \mathcal{A}_{00}(G)$ we put

$$\lambda(f, g) = \int_H a_{T,f}(h)\overline{g(h)}dh.$$

Then λ is a sesquilinear form on $\mathcal{A}_{00}(G) \times \mathcal{A}_{00}(G)$ with

$$\lambda({}_h f, g) = \lambda(f, {}_{h^{-1}} g)$$

for every $h \in H$ (see Lemma 3).

(I) First we prove that we have

$$|\lambda(f, g)| \leq \|\!|T|\!\|_p \left(\int_H \frac{|f(h)|^p}{q(h)}\right)^{1/p} \left(\int_H |g(h)|^{p'} q(h)^{p'/p}\right)^{1/p'}$$

for every $f, g \in \mathcal{A}_{00}(G)$.

Let

$$A = \left(\int_H \frac{|f(h)|^p}{q(h)}\right)^{1/p} \quad \text{and} \quad B = \left(\int_H |g(h)|^{p'} q(h)^{p'/p}\right)^{1/p'}.$$

Let $\varepsilon > 0$ and

$$0 < \delta < \min\left\{1, \frac{\varepsilon}{2\big(1 + |||T|||_p\big)\big(1 + A + B\big)}\right\}.$$

There is $\alpha \in (0, \delta)$ such that $|a^p - A^p| < \delta^p$ for every $a \in [0, \infty)$ with $|a - A| < \alpha$ and $|b^{p'} - B^{p'}| < \delta^{p'}$ for every $b \in [0, \infty)$ with $|b - B| < \alpha$. There is U_1 a relatively compact open neighborhood of e in G such that for every $x \in U_1$ we have

$$\int_H \left| a_{T,f}(xh)\overline{g(xh)} - a_{T,f}(h)\overline{g(h)} \right| dh < \delta.$$

There is also U_2 relatively compact open neighborhood of e in G such that for every $x \in U_2$

$$N_p\left(\left(\frac{f}{q^{1/p}}\right)_{x,H} - Res_H\left(\frac{f}{q^{1/p}}\right)\right) < \alpha \quad \text{and} \quad N_{p'}\Big((q^{1/p}g)_{x,H} - Res_H\big(q^{1/p}g\big)\Big) < \alpha.$$

According to Dinculeanu [34] (20.36 Proposition, p. 362) there is V open neighborhood of $\dot{e}$ in G/H $m_{G/H}$-carrable with $V \subset \omega(U_1) \cap \omega(U_2)$. We have

$$|\lambda(f, g)| \leqslant \delta + \frac{1}{m_{G/H}(V)}\left| \int_V T_H(a_{T,f}\,\overline{g})(\dot{x})dm_{G/H}(\dot{x}) \right|.$$

But

$$\begin{aligned}
&\frac{1}{m_{G/H}(V)} \int_V T_H(a_{T,f}\,\overline{g})(\dot{x})dm_{G/H}(\dot{x}) \\
&= \int_G \frac{1_{\omega^{-1}(V)}(x)\, a_{T,f}(x)}{m_{G/H}(V)^{1/p}} \frac{1_{\omega^{-1}(V)}(x)\,\overline{g(x)}q(x)}{m_{G/H}(V)^{1/p'}} dx \\
&= \left\langle M_{\frac{1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p}}}[a_{T,f}], M_{\frac{1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p'}}}[qg] \right\rangle = \left\langle M_{\frac{1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p}}} T[f], M_{\frac{1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p'}}}[qg] \right\rangle.
\end{aligned}$$

By Lemma 5

$$\frac{1}{m_{G/H}(V)} \int_V T_H(a_{T,f}\,\overline{g})(\dot{x})dm_{G/H}(\dot{x}) = \left\langle TM_{\frac{1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p}}}[f], M_{\frac{1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p'}}}[qg] \right\rangle$$

and therefore

$$\frac{1}{m_{G/H}(V)}\left|\int_V T_H(a_{T,f}\,\overline{g})(\dot{x})dm_{G/H}(\dot{x})\right| \leqslant |||T|||_p\, N_p\left(\frac{f\,1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p}}\right)N_{p'}\left(\frac{gq\,1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p'}}\right).$$

Now we show that

$$N_p\left(\frac{f\,1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p}}\right) \leqslant \delta + A.$$

We have indeed

$$N_p\left(\frac{f\,1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p}}\right)^p = \frac{1}{m_{G/H}(V)}\int_V \left(T_{H,q}|f|^p\right)(\dot{x})dm_{G/H}(\dot{x})$$

$$\leqslant \frac{1}{m_{G/H}(V)}\left|\int_V \left(T_{H,q}|f|^p\right)(\dot{x})dm_{G/H}(\dot{x}) - \int_V \left(T_{H,q}|f|^p\right)(\dot{e})dm_{G/H}(\dot{x})\right| + A^p$$

$$\leqslant \delta^p + A^p \leqslant (\delta + A)^p.$$

Similarly

$$N_{p'}\left(\frac{gq1_{\omega^{-1}(V)}}{m_{G/H}(V)^{1/p'}}\right) \leqslant \delta + B.$$

Thus

$$|\lambda(f,g)| \leqslant \frac{\varepsilon}{2} + \delta\big(A + B + 1\big)\,|||T|||_p + |||T|||_p\,AB < \varepsilon + |||T|||_p\,AB.$$

(II) Next we prove that for some $S \in CV_p(H)$ one has

$$\lambda(f,g) = \left\langle S\left[Res_H\left(\frac{f}{q^{1/p}}\right)\right], \left[Res_H\big(gq^{1/p}\big)\right]\right\rangle$$

for every $f, g \in \mathcal{A}_{00}(G)$.

By Proposition 9

$$\left\{Res_H\left(\frac{f}{q^{1/p}}\right)\middle|\, f \in \mathcal{A}_{00}(G)\right\}$$

is dense in $\mathcal{L}^p(H)$ and $\left\{Res_H\big(gq^{1/p}\big)\middle|\, g \in \mathcal{A}_{00}(G)\right\}$ is dense in $\mathcal{L}^{p'}(H)$. By (I) there is $S \in \mathcal{L}(L^p(H))$ such that

$$\lambda(f,g) = \left\langle S\left[Res_H\left(\frac{f}{q^{1/p}}\right)\right], \left[Res_H\big(gq^{1/p}\big)\right]\right\rangle$$

for every $f, g \in \mathcal{A}_{00}(G)$.

Let $f, g \in \mathcal{A}_{00}(G)$ $h \in H$ and

$$I = \left\langle S_{\,h}\left[Res_H\left(\frac{f}{q^{1/p}}\right)\right], \left[Res_H(gq^{1/p})\right]\right\rangle.$$

We have

$$_h\left(Res_H\left(\frac{f}{q^{1/p}}\right)\right) = \frac{1}{q(h)^{1/p}q(e)^{-1/p}} Res_H\left(\frac{_h f}{q^{1/p}}\right).$$

We have indeed for every $h_1 \in H$

$$\frac{f(hh_1)}{q(hh_1)^{1/p}} = \frac{1}{q(h)^{1/p}q(e)^{1/p}} \frac{f(hh_1)}{q(h_1)^{1/p}}.$$

Therefore

$$\begin{aligned} I &= \frac{1}{q(h)^{1/p}q(e)^{-1/p}} \lambda({}_h f, g) \\ &= \left\langle S\left[Res_H\left(\frac{f}{q^{1/p}}\right)\right], \left[\frac{1}{q(h)^{1/p}q(e)^{-1/p}} Res_H\left({}_{h^{-1}} g q^{1/p}\right)\right]\right\rangle. \end{aligned}$$

From

$$\frac{1}{q(h)^{1/p}q(e)^{-1/p}} Res_H({}_{h^{-1}} g q^{1/p}) = {}_{h^{-1}}\left(Res_H\left(gq^{1/p}\right)\right)$$

we get

$$I = \left\langle S\left[Res_H\left(\frac{f}{q^{1/p}}\right)\right], {}_{h^{-1}}\left[Res_H\left(gq^{1/p}\right)\right]\right\rangle = \left\langle {}_h\left(S\left[Res_H\left(\frac{f}{q^{1/p}}\right)\right]\right), \left[Res_H\left(gq^{1/p}\right)\right]\right\rangle.$$

(III) We have $i(S) = T$.

Let $f, g \in \mathcal{A}_{00}(G)$. For $x \in G$ we have ${}_x f, {}_x g \in \mathcal{A}_{00}(G)$. Using Lemma 3 we get

$$\begin{aligned} &\int_{G/H} \left\langle S\left[Res_H\left(\frac{{}_x f}{q^{1/p}}\right)\right], \left[Res_H\left({}_x g\, q^{1/p}\right)\right]\right\rangle dm_{G/H}(\dot{x}) \\ &= \int_{G/H} \lambda({}_x f, {}_x g) dm_{G/H}(\dot{x}) = \int_{G/H} T_H(a_{T,f}\, \overline{g})(\dot{x}) dm_{G/H}(\dot{x}) \\ &= \int_G a_{T,f}(x)\overline{g(x)}q(x)dx = \left\langle [a_{T,f}], [gq]\right\rangle = \left\langle T[f], [gq]\right\rangle. \end{aligned}$$

Using

$$\left(Res_H\left(\frac{{}_xf}{q^{1/p}}\right)\right) = \frac{1}{q(x)^{-1/p}q(e)^{1/p}} Res_H\left(\frac{{}_xf}{{}_xq^{1/p}}\right)$$

and

$$\frac{1}{q(x)^{-1/p}q(e)^{1/p}} Res_H\left({}_xgq^{1/p}\right) = Res_H\left(\frac{{}_xg\ {}_xq}{{}_xq^{1/p'}}\right)$$

we obtain

$$\left\langle T[f], [gq]\right\rangle = \int\limits_{G/H} \left\langle S\left[\left(\frac{f}{q^{1/p}}\right)_{x,H}\right], \left[\left(\frac{gq}{q^{1/p'}}\right)_{x,H}\right]\right\rangle dm_{G/H}(\dot{x})$$

and finally

$$\left\langle T[f], [gq]\right\rangle = \left\langle i(S)[f], [gq]\right\rangle.$$

Remarks. 1. This result is due to Lohoué ([83], [85], Théorème 5, p. 190). The special case of discrete subgroups of an unimodular locally compact group G and of $T \in PM_p(G)$ with $\operatorname{supp} T \subset H$ is in [84], Proposition p. 792. Lohoué's approach is based on the use of distributions (in the sense of Bruhat for general locally compact groups [13]). It requires the solution of the fifth Hilbert problem [95].

2. See also Anker [2], Théorème 6, p. 631 and [1], Théorème IV.7, p. 110.

7.7 First Applications: The Theorem of Kaplansky–Helson

We can now complete Theorem 1 and Corollary 2 of Sect. 7.5. We recall that for $g \in \mathcal{L}^\infty(\widehat{H})$ we have $sp((g\circ\tau\circ\gamma)^{\cdot}) \subset H$. Conversely the following result is verified.

Theorem 1. *Let G be a locally compact abelian group, H a closed subgroup and $f \in \mathcal{L}^\infty(\widehat{G})$ with $sp(\dot{f}) \subset H$. Then there is $g \in \mathcal{L}^\infty(\widehat{H})$ such that $f(\chi) = g(\tau(\gamma(\chi)))$ $m_{\hat{G}}$-locally almost everywhere on $\widehat{G}$.*

Proof. Let $T = \Lambda_{\hat{G}}(\dot{f})$, then $T \in CV_2(G)$, $f \in \widehat{T}$ and $sp(\dot{f}) = (\operatorname{supp} T)^{-1}$. By Theorem 10 of Sect. 7.6 there is $S \in CV_2(H)$ with $i(S) = T$. Let $g \in \widehat{S}$. By Theorem 1 Sect. 7.5 we have $f(\chi) = g(\tau(\gamma(\chi)))$ $m_{\hat{G}}$-locally almost everywhere on $\widehat{G}$.

Remark. This result is due to Reiter ([103], Lemma 1.2., p. 554). For $f \in C^b(\widehat{G})$ see [102], Theorem 2, p. 257. See also Reiter and Stegeman ([105], Corollary 7.3.13).

Theorem 2. *Let G be a locally compact abelian group and H a closed subgroup. Then $\dot{g} \mapsto (g\circ\tau\circ\gamma)^{\cdot}$ is an isometric Banach algebra isomorphism of $L^\infty(\widehat{H})$ onto $\left\{f \in L^\infty(\widehat{G}) \middle| spf \subset H\right\}$.*

Remark. Using invariant means on $H^{\perp}$, Herz ([55], 5.12. Theorem, p. 220) 5.12. Theorem, p. 220) and Gilbert ([49], 4.1. Lemma, p. 84) proved the existence of a projection P of $L^{\infty}(\widehat{G})$ onto $L^{\infty}(\widehat{H})$ such that

$$P(f * u) = (T_{H^{\perp}} f)\circ\tau^{-1} * P(u)$$

for every $f \in L^{1}(\widehat{G})$ and every $u \in L^{\infty}(\widehat{G})$. Even for $G = \mathbb{R}$ and $H = \mathbb{Z}$ the result is deep: P is a periodization map of $L^{\infty}(\mathbb{R})$ onto $L^{\infty}(\mathbb{T})$. Let now G be a noncommutative locally compact group and H a closed subgroup. Suppose that H satisfies one of the following conditions: H is normal or open or amenable. Then by [32] (Theorem 15) there is a projection P of $CV_p(G)$ onto $CV_p(H)$ with

$$P(uT) = Res_H u P(T)$$

for every $u \in A_p(G)$. However there is no projection of $CV_2(SL(2,\mathbb{R}))$ onto $CV_2(SL(2,\mathbb{Z}))$ ([3], p. 382, Remark 2.)

Theorem 3. *Let G be a locally compact abelian group, H a closed subgroup,* $1 < p < \infty$, *$g \in \mathbb{C}^{\hat{H}}$ and $f = g\circ\tau\circ\gamma$. Then the following properties are equivalent:*

1. *There is $S \in CV_p(H)$ with $\dot{g} = \widehat{S}$,*
2. *There is $T \in CV_p(G)$ with $\dot{f} = \widehat{T}$.*

Proof. 1. Implies 2. indeed, according to Theorem 1 Sect. 7.5, $\dot{f} = \widehat{i(S)}$.

2. Implies 1.

We have $sp\dot{f} = (\operatorname{supp} T)^{-1}$ and $g \in \mathcal{L}^{\infty}(\widehat{H})$. By Corollary 2 Sect. 7.5 $sp\dot{f} = sp((g\circ\tau\circ\gamma)^{\dot{}}) = sp(\dot{g})$. Consequently $\operatorname{supp} T \subset H$. By Theorem 10 of Sect. 7.6 there is $S \in CV_p(H)$ with $i(S) = T$. Let $g' \in \widehat{S}$, by Theorem 1 Sect. 7.5 $f(\chi) = g'(\tau(\gamma(\chi)))$ $m_{\hat{G}}$-locally almost everywhere on $\widehat{G}$. It follows that $g(\chi) = g'(\chi)$ $m_{\hat{H}}$-locally almost everywhere on $\widehat{H}$. Then $\dot{g} = \widehat{S}$.

Remarks. 1. For $G = \mathbb{R}$ and $H = \mathbb{Z}$ Theorem 3 is due to de Leeuw [74] (Theorem 4.5, p. 377).

2. Theorem 3 is due to Saeki [108] (Corollary 3.5, p. 417). This result was also obtained by Lohoué in [80] for $g \in C(\widehat{H})$ (Théorème I.2, p. 20) but for a far more general situation: instead of the inclusion of a closed subgroup H into G, Lohoué studied the continuous homomorphisms of a locally compact abelian group G into a locally compact group H. For related results see also Lust-Piquard [89]. The non commutative case has been recently investigated by Delmonico [26]. For G a Lie group, Delmonico associated to every element of the Lie algebra a linear continuous map of $CV_p(\mathbb{R})$ into $CV_p(G)$.

Lemma 4. *Let G be a locally compact group,* $1 < p < \infty$ *and $T \in CV_p(G)$. Then for every $a \in G$ we have* $\operatorname{supp}(\lambda_G^p(\delta_a)T) = (\operatorname{supp} T)a^{-1}$.

Proof. Let $x \notin (\operatorname{supp} T)a^{-1}$. There is U, neighborhood of e and V, neighborhood of xa with $\langle T[\varphi], [\psi]\rangle = 0$ for every $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset U$ and $\operatorname{supp}\psi \subset V$. Let W be a compact neighborhood of x with $W \subset Va^{-1}$. Let now $\varphi, \psi \in C_{00}(G)$ with $\operatorname{supp}\varphi \subset U$ and $\operatorname{supp}\psi \subset W$. Then

$$\left\langle \lambda_G^p(\delta_a)T[\varphi], [\psi]\right\rangle = \left\langle T[\varphi], \lambda_G^{p'}(\delta_{a^{-1}})[\psi]\right\rangle = \left\langle T[\varphi], [\psi_{a^{-1}}\Delta_G(a^{-1})^{1/p'}]\right\rangle$$

but $\operatorname{supp}\psi_{a^{-1}}\Delta_G(a^{-1})^{1/p'} \subset V$ and therefore $\left\langle \lambda_G^p(\delta_a)T[\varphi], [\psi]\right\rangle = 0$, consequently $x \notin \operatorname{supp}(\lambda_G^p(\delta_a)T)$. This proves that $\operatorname{supp}(\lambda_G^p(\delta_a)T) \subset (\operatorname{supp} T)a^{-1}$ consequently $\operatorname{supp} T = \operatorname{supp}\lambda_G^p(\delta_{a^{-1}}\lambda_G^p(\delta_a)T) \subset \left(\operatorname{supp}\lambda_G^p(\delta_a)T\right)a$ hence $(\lambda_G^p(\delta_a)T) = (\operatorname{supp} T)a^{-1}$.

As a main consequence of Theorem 10 of Sect. 7.6 we obtain the following generalization of the theorem of Kaplansky–Helson.

Theorem 5. *Let G be a locally compact group, $1 < p < \infty$, $a \in G$ and $T \in CV_p(G)$ with $\operatorname{supp} T = \{a\}$. Then there is $\alpha \in \mathbb{C}$ with $\alpha \neq 0$ and $T = \alpha\lambda_G^p(\delta_{a^{-1}})$.*

Proof. Suppose first $a = e$. By Theorem 10 of Sect. 7.6 there is $S \in CV_p(H)$ with $i(S) = T$ and $H = \{e\}$. There is $\alpha \in \mathbb{C}$ with $\alpha \neq 0$ and $S = \alpha\, \mathrm{id}_{L^p_{\mathbb{C}}(H)}$. Clearly for $\varphi, \psi \in C_{00}(G)$ and $x \in G$ we have

$$\left\langle [\varphi_{x,H}], [\psi_{x,H}]\right\rangle = \varphi(x)\overline{\psi(x)}$$

this implies

$$\left\langle T[\varphi], [\psi]\right\rangle = \left\langle i(S)[\varphi], [\psi]\right\rangle = \alpha \int_G \varphi(x)\overline{\psi(x)}dx = \alpha\left\langle [\varphi], [\psi]\right\rangle$$

and therefore $T = \alpha\, \mathrm{id}_{L^p_{\mathbb{C}}(G)}$.

In the general case we have $\operatorname{supp}(\lambda_G^p(\delta_a)T) = \{e\}$ according to Lemma 4. There is $\alpha \in \mathbb{C}$ with $\alpha \neq 0$ and $\lambda_G^p(\delta_a)T = \alpha\, \mathrm{id}_{L^p_{\mathbb{C}}(G)}$, and therefore $T = \alpha\lambda_G^p(\delta_{a^{-1}})$.

Remarks. 1. Theorem 5 is due to Herz ([61], p. 101).
2. Considering the case $\operatorname{supp} T = \emptyset$, we also obtain a second proof of Theorem 3 of Sect. 6.3.
3. The above proof is essentially due to Anker ([1], Remarque 3) p. 111).
4. For $G = \mathbb{R}$ and $p = 2$ this result is due to Beurling [5]. See the clarifying discussion in [52].
5. For $p = 2$ and G a general locally compact abelian group, the theorem has been obtained by Kaplansky, using the structure theory of locally compact abelian groups ([71], Applications: (1), p. 135) and by Helson without the use of this structure theory ([53], Theorem 1, p. 500).
6. For $p = 2$ and G a general locally compact group, the theorem has been obtained by Eymard ([41] (4.9.) Théorème, p. 229).

We are now able to give a description of the convolution operators with finite support.

Theorem 6. *Let G be a locally compact group, $1 < p < \infty$ and $T \in CV_p(G)$. Suppose that* $\operatorname{supp} T = \{a_1, \ldots, a_n\}$ *with $n > 1$ and $a_1, \ldots, a_n$ distincts. Then there is $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with $\alpha_1 \neq 0, \ldots, \alpha_n \neq 0$ and*

$$T = \sum_{j=1}^{n} \alpha_j \lambda_G^p(\delta_{a_j^{-1}}).$$

Proof. Let $U_1, \ldots, U_n, V_1, \ldots, V_n$ be open subsets of G with $a_j \in V_j$, $\overline{V_j} \subset U_j$, U_j relatively compact for $1 \leq j \leq n$, $U_i \cap U_j = \emptyset$ for $i \neq j$. For $1 \leq j \leq n$ there is also $\tau_j \in A_p(G) \cap C_{00}(G; \mathbb{R})$ with $\tau_j(x) = 1$ on $\overline{V_j}$ and $\operatorname{supp} \tau_j \subset U_j$. By Theorem 2 of Sect. 6.2 we have $\operatorname{supp} \tau_j T = \{a_j\}$. Let v be an arbitrary element of $A_p(G)$. Then $v - \sum_{j=1}^{n} \tau_j v = 0$ on $V_1 \cup \ldots \cup V_n$, by Proposition 4 of Sect. 6.3 this implies

$$\left(v - \sum_{j=1}^{n} \tau_j v\right) T = 0 \quad \text{and} \quad v\left(T - \sum_{j=1}^{n} \tau_j T\right) = 0,$$

consequently

$$\operatorname{supp}\left(T - \sum_{j=1}^{n} \tau_j T\right) = \emptyset \quad \text{i.e.} \quad T = \sum_{j=1}^{n} \tau_j T.$$

Theorem 5 permits to finish the proof.

7.8 A Restriction Property for A_p

Theorem 1. *Let G be a locally compact abelian group and H a closed subgroup. Then for every $u \in A_2(G)$ we have $Res_H u \in A_2(H)$ and $\|Res_H u\|_{A_2} \leqslant \|u\|_{A_2}$.*

Proof. We prove at first that for $f \in C_{00}(\widehat{G})$ we have

$$\Phi_{\hat{H}}\left(\left[\left(T_{H^\perp} f\right) \circ \tau^{-1}\right]\right) = Res_H\left(\Phi_{\hat{G}}([f])\right)$$

(we use the notations of Sect. 7.5).

For $h \in H$ by definition

$$\Phi_{\hat{H}}\left(\left[\left(T_{H^\perp} f\right)\circ\tau^{-1}\right]\right)(h) = \int_{\hat{H}} \left(T_{H^\perp} f\right)(\tau^{-1}(\chi))\chi(h)d\chi$$

$$= \int_{\hat{G}/H^\perp} \left(T_{H^\perp} f\right)(\dot{\chi})\tau(\dot{\chi})(h)d\dot{\chi} = \int_{\hat{G}/H^\perp} \left(T_{H^\perp} f\right)(\dot{\chi})\sigma(h)(\dot{\chi})d\dot{\chi}$$

$$= \int_{\hat{G}} f(\chi)\chi(h)d\chi = \Phi_{\hat{G}}([f])(h).$$

Let $u \in A_2(G)$, $f = \Phi_{\hat{G}}^{-1}(u)$, $g \in f$ and (g_n) a sequence of $C_{00}(\widehat{G})$ with $N_1(g - g_n) \to 0$. For every $n \in \mathbb{N}$ we have

$$\left\| Res_H u - \Phi_{\hat{H}}\left(\left(T_{H^\perp} f\right)\circ\tau^{-1}\right)\right\|_\infty \leq \left\| Res_H\left(\Phi_{\hat{G}}([f])\right) - Res_H\left(\Phi_{\hat{G}}([g_n])\right)\right\|_\infty$$
$$+\left\|\left(\Phi_{\hat{H}}\left([T_{H^\perp} g_n]\circ\tau^{-1}\right)\right) - \left(\Phi_{\hat{H}}\left([T_{H^\perp} f]\circ\tau^{-1}\right)\right)\right\|_\infty \leq 2N_1(g - g_n).$$

Consequently

$$Res_H u = \Phi_{\hat{H}}\left(\left(T_{H^\perp}\Phi_{\hat{G}}^{-1}(u)\right)\circ\tau^{-1}\right).$$

We generalize now this result to every $1 < p < \infty$ and to every locally compact group.

Theorem 2. *Let G be an arbitrary locally compact group, H a closed subgroup and $1 < p < \infty$. Then for every $u \in A_p(G)$ we have $Res_H u \in A_p(H)$ and $\|Res_H u\|_{A_p(H)} \leqslant \|u\|_{A_p(G)}$.*

Proof. To begin with we prove that if $u = \overline{k} * \check{l}$ where $k, l \in C_{00}(G)$ then $Res_H u \in A_p(H)$ and $\|Res_H u\|_{A_p} \leqslant N_p(k)N_{p'}(l)$.

Let $r = \tau_p k$ and $s = \tau_{p'} l$. For $h \in H$ we have

$$u(h) = \left\langle \lambda_G^{p'}(\delta_h)[s], [r]\right\rangle = \Delta_G(h)^{1/p'} \int_{G/H} \left(\int_H \frac{s_h(yh')\overline{r(yh')}dh'}{q(yh')}\right) dm_{G/H}(\dot{y})$$
$$= \int_{G/H} \left\langle \lambda_H^{p'}(\delta_h)\left[\left(\frac{s}{q^{1/p'}}\right)_{y,H}\right], \left[\left(\frac{r}{q^{1/p}}\right)_{y,H}\right]\right\rangle dm_{G/H}(\dot{y}).$$

For $\dot{y} \in G/H$ let

$$(L(\dot{y}))(h) = \left\langle \lambda_H^{p'}(\delta_h)\left[\left(\frac{s}{q^{1/p'}}\right)_{y,H}\right], \left[\left(\frac{r}{q^{1/p}}\right)_{y,H}\right]\right\rangle$$

then $L(\dot{y}) = \overline{f(y)} *_H g(y)^{\vee}$ where

$$f(y) = \tau_p\left(\left(\frac{r}{q^{1/p}}\right)_{y,H}\right) \quad \text{and} \quad g(y) = \tau_{p'}\left(\left(\frac{s}{q^{1/p'}}\right)_{y,H}\right).$$

We claim that L is a continuous map of G/H into $A_p(H)$.

Let $y_0 \in G$ and $\varepsilon > 0$. We may (see Lemma 8, Sect. 7.1) choose U, open neighborhood of y_0 in G, such that for every $y \in U$

$$N_p\left(\left(\frac{r}{q^{1/p}}\right)_{y,H} - \left(\frac{r}{q^{1/p}}\right)_{y_0,H}\right) < \eta \quad \text{and} \quad N_{p'}\left(\left(\frac{s}{q^{1/p'}}\right)_{y,H} - \left(\frac{s}{q^{1/p'}}\right)_{y_0,H}\right) < \eta$$

where

$$0 < \eta < \min\left\{1, \frac{\varepsilon}{\left(1 + N_p\left(\left(\frac{r}{q^{1/p}}\right)_{y_0,H}\right) + N_{p'}\left(\left(\frac{s}{q^{1/p'}}\right)_{y_0,H}\right)\right)}\right\}.$$

For every $\dot{y} \in \omega(U)$ we then have $\|L(\dot{y}) - L(\dot{y}_0)\|_{A_p} < \varepsilon$.

On the other hand

$$\int_{G/H}^{*} \|L(\dot{y}))\|_{A_p} dm_{G/H}(\dot{y}) \leqslant \int_{G/H}^{*} N_p(f(y))\, N_{p'}(g(y)) dm_{G/H}(\dot{y})$$

$$\leqslant \left(\int_{G/H}^{*} N_p(f(y))^p dm_{G/H}(\dot{y})\right)^{1/p} \left(\int_{G/H}^{*} N_p(g(y))^{p'} dm_{G/H}(\dot{y})\right)^{1/p'} = N_p(k)\, N_{p'}(l).$$

Then $L \in \mathcal{L}^1(G/H; A_p(H), m_{G/H})$. Let $v = \int_{G/H} L(\dot{y}) dm_{G/H}(\dot{y})$, we have $v \in A_p(H)$ (see Sect. 3.3). For every $h \in H$ $v(h) = \int_{G/H} L(\dot{y})(h) dm_{G/H}(\dot{y}) = u(h)$ and therefore $Res_H u \in A_p(H)$ with $\|Res_H u\|_{A_p} \leqslant N_p(k) N_{p'}(l)$.

Next for $k_1, \ldots, k_n$, $l_1, \ldots, l_n \in C_{00}(G)$ and $u = \sum_{j=1}^{n} \overline{k}_j * \check{l}_j$ we get $Res_H u \in A_p(H)$ and

$$\|Res_H u\|_{A_p} \leqslant \sum_{j=1}^{n} N_p(k_j) N_{p'}(l_j).$$

Finally let u be an arbitrary element of $A_p(G)$. Let $\varepsilon > 0$. By Proposition 6 of Sect. 3.1 there is a sequence (k_n) and a sequence (l_n) of $C_{00}(G)$ with $u = \sum_{n=1}^{\infty} \overline{k}_n * \check{l}_n$ and

$$\sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) < \|u\|_{A_p} + \varepsilon.$$

Let for every $n \in \mathbb{N}$ $s_n = \sum_{j=1}^{n} \overline{k}_j * \check{l}_j$. One has $Res_H s_n \in A_p(H)$. For $n' > n$ we also have

$$\|Res_H s_{n'} - Res_H s_{n'}\|_{A_p} \leqslant \sum_{j=n+1}^{n'} N_p(k_j) N_{p'}(l_j).$$

There is consequently $v \in A_p(H)$ with $\lim \|v - Res_H s_n\|_{A_p} = 0$. But $\lim \|u - s_n\|_\infty = 0$ and therefore $v = Res_H u$, this implies $Res_H u \in A_p(H)$ and $\|Res_H u\|_{A_p} = \lim \|Res_H s_n\|_{A_p}$. For every $n \in \mathbb{N}$ we have

$$\|Res_H s_n\|_{A_p} \leqslant \sum_{j=1}^{\infty} N_p(k_j) N_{p'}(l_j)$$

consequently $\|Res_H u\|_{A_p} < \|u\|_{A_p} + \varepsilon$.

Remarks. 1. This result is due to Herz ([57], p. 244 and [61], Theorem 1 a, p. 92 and p. 107). We have followed [23]

2. By Theorem 4 of Sect. 3.3 we have $v(Res_H u) \in A_p(H)$ and

$$\|v(Res_H u)\|_{A_p(H)} \leqslant \|v\|_{A_p(H)} \|u\|_{A_p(G)}$$

for every $v \in A_p(H)$. Theorem 2 is a much stronger result.

Scholium 3. *Let G be an arbitrary locally compact group, H a closed subgroup and $1 < p < \infty$. For every $r, s \in C_{00}(G)$ and every $y \in G$ we put*

$$f(y) = \tau_p\left(\left(\frac{r}{q^{1/p}}\right)_{y,H}\right) \quad \textit{and} \quad g(y) = \tau_{p'}\left(\left(\frac{s}{q^{1/p'}}\right)_{y,H}\right).$$

Then:

1. $\int_{G/H} \overline{f(y)} *_H g(y)\check{}\, dm_{G/H}(\dot{y}) \in A_p(H)$,
2. $Res_H(\overline{r} *_G \check{s}) = \int_{G/H} \overline{f(y)} *_H g(y)\check{}\, dm_{G/H}(\dot{y})$.

Theorem 4. *Let G be an arbitrary locally compact group, H a closed subgroup and $1 < p < \infty$, and $S \in PM_p(H)$, then:*

1. $i(S) \in PM_p(G)$,
2. $\langle u, i(S)\rangle_{A_p,PM_p} = \langle Res_H u, S\rangle_{A_p,PM_p}$ *for* $u \in A_p(G)$.

Proof. We put

$$F(u) = \langle Res_H u, S\rangle_{A_p,PM_p}$$

for every $u \in A_p(G)$. By Theorem 2 $F \in A_p(G)'$, by Theorem 6 of Sect. 4.1, there is $T \in PM_p(G)$ with

$$F(u) = \langle u, T\rangle_{A_p,PM_p}$$

for every $u \in A_p(G)$. Now let $r, s \in C_{00}(G)$, we have

$$\left\langle(\overline{\tau_p r}) * (\tau_{p'} s)\check{}, T\right\rangle_{A_p,PM_p} = \overline{\left\langle T[r],[s]\right\rangle} = \left\langle Res_H\left((\overline{\tau_p r}) * (\tau_{p'} s)\check{}\right), S\right\rangle_{A_p,PM_p}.$$

But according to Scholium 3

$$\overline{\left\langle Res_H\left(\overline{\tau_p r} * (\tau_{p'} s)\check{}\right), S\right\rangle}_{A_p,PM_p}$$
$$= \int_{G/H} \left\langle S\left[\left(\frac{r}{q^{1/p}}\right)_{y,H}\right], \left[\left(\frac{s}{q^{1/p'}}\right)_{y,H}\right]\right\rangle dm_{G/H}(\dot{y}) = \left\langle i(S)[r],[s]\right\rangle.$$

This implies $T = i(S)$ and consequently $T \in PM_p(G)$.

In full analogy with the case of $L^1(G)$ (see [105], p. 231), Herz proved the following Theorem ([57], Théorème p. 244 and [61], Theorem 1b, p. 92).

Theorem 5. *Let G be a locally compact group, $1 < p < \infty$ and H a closed subgroup of G. Then Res_H maps $A_p(G)$ onto $A_p(H)$. We also have:*

1. *For every $u \in A_p(G)$*

$$\|Res_H u\|_{A_p(H)} = \inf\left\{\|u + v\|_{A_p(G)} \,\middle|\, v \in A_p(G), Res_H v = 0\right\},$$

2. *For every $u \in A_p(H)$ and for every $\varepsilon > 0$ there is $v \in A_p(G)$ with $Res_H v = u$ and*

$$\|v\|_{A_p(G)} \leqslant (1+\varepsilon)\|u\|_{A_p(H)}.$$

Proof. For every $u \in A_p(G)$ we put $\varphi(u) = Res_H u$. According to Theorem 2 φ is a linear contraction of $A_p(G)$ into $A_p(H)$. For $S \in PM_p(H)$ we have

$${}^t\varphi(\Psi_H^p(S)) = \Psi_G^p(i(S))$$

where Ψ_G^p is defined in Sect. 4.1. From Theorem 2 of Sect. 7.3 it follows that ${}^t\varphi$ is an isometry of $A_p(H)'$ into $A_p(G)'$ and therefore that ${}^t\varphi(A_p(H)')$ is closed in $A_p(G)'$. Consequently $\varphi(A_p(G)) = A_p(H)$. By 1 Lemma p. 487 of [38] there is $K > 0$ such that for every $u \in A_p(H)$ there is $v \in A_p(G)$ with $\varphi(v) = u$ and

$$\|v\|_{A_p(G)} \leqslant K\|u\|_{A_p(H)}.$$

Let $W_H = \left\{T \in PM_p(G) \middle| \Psi_G^p(T)(u) = 0 \text{ for every } u \in Ker\varphi\right\}$. Then we have $i(PM_p(H)) = W_H$.

Indeed for $S \in PM_p(H)$, the relation

$$\Psi_G^p(i(S))(u) = \Psi_H^p(S)(\varphi(u)),$$

implies $i(S) \in W_H$. Conversely let $T \in W_H$. For every $u \in A_p(H)$ we set

$$f(u) = \Psi_G^p(T)(v)$$

where $v \in A_p(G)$ with $Res_H v = u$ and $\|v\|_{A_p(G)} \leqslant K\|u\|_{A_p(H)}$. Then $f \in A_p(H)'$ and consequently there is $S \in PM_p(H)$ with $f = \Psi_H^p(S)$. We obtain $T = i(S)$.

Let π be the canonical map of $A_p(G)$ onto $A_p(G)/Ker\varphi$. For $S \in PM_p(H)$ we put

$$\varepsilon(S)(\pi(u)) = \Psi_G^p(i(S))(u)$$

for every $u \in A_p(G)$. Then ε is an isometry of $PM_p(H)$ onto $(A_p(G)/Ker\varphi)'$. This finally implies (1) and (2).

Remarks. 1. The fact that for every $u \in A_p(H)$ there is a $v \in A_p(G)$ with $Res_H v = u$, was also obtained, independently of Herz, by McMullen in [93], p. 47 (4.21) Theorem.

2. For the above proof, see Delaporte and Derighetti [23].
3. For G and H both unimodular, assuming moreover that H is amenable, Fiorillo obtained recently a much stronger extension theorem [48].

7.9 Subgroups as Sets of Synthesis

Definition 1. Let G be a locally compact group and $1 < p < \infty$. A closed subset F of G is said to be a set of p-synthesis in G if for every $u \in A_p(G)$ with $Res_F u = 0$ and for every $\varepsilon > 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$ and with $\|u - v\|_{A_p} < \varepsilon$.

Theorem 1. *Let G be a locally compact group and $1 < p < \infty$. Then:*

1. *The empty set is a set of p-synthesis in G,*
2. *Every finite subset of G is a set of p-synthesis in G.*

Proof. The statement (1) results from Corollary 7 of Sect. 3.1. Let $a_1, \ldots, a_n$ be distinct elements of G, $u \in A_p(G)$ with $u(a_j) = 0$ for $1 \leq j \leq n$ and $\varepsilon > 0$. We will prove the second statement by using a slight modification of the proof of Theorem 2 of Sect. 4.3 (which is in fact the statement for $n = 1$).

There is $w \in A_p(G) \cap C_{00}(G)$ with

$$\|u - w\|_{A_p} < \frac{\varepsilon}{6n^{1/p}}$$

and V an open neighborhood of e in G such that for every $x \in V$

$$\|w - w_x\|_{A_p} < \frac{\varepsilon}{6n^{1/p}}.$$

Let W be an open relatively compact neighborhood of e in G with $W \subset V$ and

$$a_j W \cap a_h W = \emptyset$$

for $1 \leq j, h \leq n$ with $j \neq h$. Let K be compact subset of W with $m(K) > m(W)/2$. Consider

$$k = \frac{1_K}{m(K)}, \quad l = \sum_{j=1}^{n} w 1_{a_j W} \quad \text{and} \quad v = w * \check{k} - l * \check{k}.$$

We have $\|u - v\|_{A_p} < \varepsilon/6 + \|w - v\|_{A_p}$ and $\|w - v\|_{A_p} \leq \|w - w * \check{k}\|_{A_p} + \|l * \check{k}\|_{A_p}$ with $\|w - w * \check{k}\|_{A_p} < \varepsilon/6$. We have

$$N_{p'}(k) = \frac{1}{m(K)^{1/p}} < \frac{2}{m(W)^{1/p}}$$

and

$$N_p(l)^p = \sum_{j=1}^{n} \int_{a_j W} |w(x)|^p dx.$$

Taking in account that

$$|w(y)| < \frac{\varepsilon}{3n^{1/p}}$$

for $1 \leq j \leq n$ and $y \in a_j W$ we get

$$N_p(l) \leq \frac{\varepsilon}{3} m(W)^{1/p}$$

and $\|u - v\|_{A_p} < \varepsilon$. Choose finally Z open neighborhood of e in G with $ZK \subset W$, it suffices to verify that the function v vanishes on

$$\bigcup_{j=1}^{n} a_j Z$$

to conclude the proof.

Lemma 2. *Let G be a locally compact group, F a closed subset of G and* $1 < p < \infty$. *Then the following statements are equivalent:*

1. *F is a set of p-synthesis in G,*
2. *For every* $u \in A_p(G)$ *with* $Res_F u = 0$ *and for every* $T \in PM_p(G)$ *with* $\operatorname{supp} T \subset F$ *we have* $\langle u, T \rangle_{A_p, PM_p} = 0$.

Proof. Suppose that F is a set of p -synthesis in G. Let $\varepsilon > 0$. There is $v \in A_p(G) \cap C_{00}(G)$ with

$$\|u - v\|_{A_p} < \frac{\varepsilon}{\left(1 + \|T\|_p\right)}$$

and $\operatorname{supp} v \cap F = \emptyset$. We obtain $\left|\langle u, T \rangle_{A_p, PM_p}\right| < \varepsilon + \left|\langle v, T \rangle_{A_p, PM_p}\right|$. Proposition 7 of Sect. 6.3 implies $\left|\langle u, T \rangle_{A_p, PM_p}\right| < \varepsilon$. We get therefore $\langle u, T \rangle_{A_p, PM_p} = 0$.

To prove the converse, assume that the set F is not of p-synthesis. Let B the closure in $A_p(G)$ of $\left\{v \middle| v \in A_p(G) \cap C_{00}(G), \operatorname{supp} v \cap F = \emptyset\right\}$. Then there is $u \in A_p(G) \setminus B$ with $Res_F u = 0$. By Theorem 6 of Sect. 4.1 and the theorem of Hahn–Banach there is $T \in PM_p(G)$ with $\langle u, T \rangle_{A_p, PM_p} \neq 0$ and $\langle v, T \rangle_{A_p, PM_p} = 0$ for every $v \in B$. The last condition and Theorem 1 of Sect. 6.2 imply that $\operatorname{supp} T \subset F$. The existence of u contradicts (2).

The fact that every closed subgroup of a locally compact abelian group is a set of synthesis, is a famous result due to Reiter [104] (Corollary p. 146) and [102], Theorem 4, p. 259. In this paragraph we investigate whether the corresponding property is verified for noncommutative groups.

Lemma 3. *Let G be a locally compact group, H a closed subgroup of G and* $1 < p < \infty$. *Then the following statements are equivalent:*

1. *H is a set of p-synthesis in G,*
2. *For every* $S \in CV_p(H)$ *with* $i(S) \in PM_p(G)$ *we have* $S \in PM_p(H)$.

Proof. Assume first that H is a set of p -synthesis in G. Let $S \in CV_p(H)$ with $i(S) \in PM_p(G)$. Taking in account Theorem 5 of Sect. 7.8, Lemma 2 and the relation $\operatorname{supp} i(S) = \operatorname{supp} S$ (Theorem 3 of Sect. 7.4) we put for $u \in A_p(H)$ $f(u) = \langle v, i(S) \rangle_{A_p, PM_p}$ with $v \in A_p(G)$ such that $Res_H v = u$ and $\|v\|_{A_p} \leq 2\|u\|_{A_p}$. From $f \in A_p(H)'$ it follows the existence of $S_1 \in PM_p(H)$ with $f(u) = \langle u, S_1 \rangle_{A_p, PM_p}$ for every $u \in A_p(H)$.

Let v be an arbitrary element of $A_p(G)$. We have $\langle v, i(S) \rangle_{A_p, PM_p} = f(Res_H v) = \langle Res_H v, S_1 \rangle_{A_p, PM_p}$. But $\langle Res_H v, S_1 \rangle_{A_p, PM_p} = \langle v, i(S_1) \rangle_{A_p, PM_p}$, consequently $i(S) = i(S_1)$, $S = S_1$ then $S \in PM_p(H)$.

Suppose now that the condition (2) is satisfied. Let u be an element of $A_p(G)$ with $Res_H u = 0$ and $T \in PM_p(G)$ with $\operatorname{supp} T \subset H$. It suffices to verify, according to Lemma 2, that $\langle u, T\rangle_{A_p,PM_p} = 0$. By Theorem 10 of Sect. 7.6 there is $S \in CV_p(H)$ with $i(S) = T$. By our hypothesis and by Theorem 4 of Sect. 7.8 $\langle u, T\rangle_{A_p,PM_p} = \langle u, i(S)\rangle_{A_p,PM_p} = \langle Res_H u, S\rangle_{A_p,PM_p} = 0$.

The following is a generalization of Reiter's result.

Theorem 4. *Let G be a locally compact group and H an unimodular closed subgroup of G. Then H is a set of 2-synthesis in G.*

Proof. According to the approximation theorem (Remark 2 after Definition 2 of Sect. 4.1) $PM_2(H) = CV_2(H)$. The preceding lemma permits to conclude.

Remark. If we admit the approximation theorem for all locally compact groups, then Theorem 4 is verified for every closed subgroup of G.

For $p \neq 2$ the situation is more delicate!

Theorem 5. *Let G be a locally compact group and H a closed amenable subgroup of G. Then for every $1 < p < \infty$ H is a set of p-synthesis in G.*

Proof. By Corollary 3 of Sect. 5.4 we have $CV_p(H) = PM_p(H)$. As above Lemma 3 permits to finish the proof.

Remarks. 1. Theorems 4 and 5 are due to Herz [61]. There are no general results for nonamenable subgroups. However Herz proved in [61] (Proposition 1, p. 92) that normal or open subgroups are sets of p -synthesis. Delaporte and Derighetti obtained the p -synthesis for a larger class of nonamenable subgroups in [24] (Corollary 4, p. 1432).

2. Herz introduced in [61] the following notion: a closed subset F of G is said to be of local p-synthesis if for every $u \in A_p(G) \cap C_{00}(G)$ with $Res_H u = 0$ and for every $\varepsilon > 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$ and with $\|u - v\|_{A_p} < \varepsilon$. If every element u of $A_p(G)$ belongs to the closure of $uA_p(G)$ in $A_p(G)$, then clearly every set of local synthesis is a set of synthesis. Consequently (see Theorem 4 of $Sect.$ 5.4) in amenable groups every set of local synthesis is of synthesis. Herz proved in [61] that every closed subgroup, of an arbitrary locally compact group, is a set of local p-synthesis. For complementary results see Lohoué [82] and Derighetti [30].
3. The question whether, for a general locally compact group G, every u of $A_p(G)$ belongs to the closure of $uA_p(G)$ is presently out of reach, even for $p = 2$. See the notes to Chap. 5 for a list of nonamenable groups including $SL_2(\mathbb{R})$ having this property for $p = 2$.
4. If for the closed subgroup H of G $PM_p(H) = CV_p(H)$ then H is a set of p-synthesis in G.

Chapter 8
$CV_p(G)$ as a Subspace of $CV_2(G)$

We extend to amenable groups the relations $CV_{p(G)} \subset CV_2(G)$ and $A_2(G) \subset A_p(G)$.

8.1 A Canonical Map of $\mathcal{L}(L^p(X,\mu))$ into $\mathcal{L}(L^p_{\mathcal{H}}(X,\mu))$

Theorem 1. *There is:*

1. *A Hausdorff space* Z*,*
2. *A probability measure* ν *on* Z*,*
3. *A sequence* (h_n) *of complex continuous functions on* Z*, such that*
    1. $h_n \in \mathcal{L}^p(Z,\nu)$ *for every* n *and for every* $1 \leq p < \infty$*,*
    2. $\int_Z h_m(\gamma)\overline{h_n(\gamma)}d\nu(\gamma) = \delta_{m,n}$ *for every* m, n*,*
    3. $$\left(\int_Z \left|\sum_{n=1}^N \alpha_n h_n(\gamma)\right|^p d\nu(\gamma)\right)^{1/p} = \left(\sum_{n=1}^N |\alpha_n|^2\right)^{1/2} \Gamma\left(\frac{p+2}{2}\right)^{1/p}$$

    for every $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ *and every* $1 \leq p < \infty$*.*

Proof. For every $n \in \mathbb{N}$ we put $Z_n = \mathbb{C}$, $Z = \prod_{n=1}^{\infty} Z_n$ with the product topology and $g_n(\gamma) = \gamma(n)$ for every $\gamma \in Z$. We define the Radon measure ν_n on Z_n by

$$\nu_n(\varphi) = \frac{1}{2\pi} \int\int_{\mathbb{R}\times\mathbb{R}} \varphi(x+iy)e^{-(x^2+y^2)/2}dxdy$$

for $\varphi \in C_{00}(Z_n)$.

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Consider on

$$\mathcal{F} = \left\{(\beta_1, \dots, \beta_n) \middle| n \in \mathbb{N},\ \beta_1 \cdots, \beta_n \in \mathbb{N},\ \beta_1 < \cdots < \beta_n\right\}$$

the partial order defined by the inclusion.

For every $J = (\beta_1, \dots, \beta_n) \in \mathcal{F}$ we put:

$$Z_J = Z_{\beta_1} \times \cdots \times Z_{\beta_n},$$
$$\nu_J = \nu_{\beta_1} \otimes \cdots \otimes \nu_{\beta_n}$$

and

$$p_J(\gamma) = (\gamma(\beta_1), \dots, \gamma(\beta_n)) \text{ for every } \gamma \in Z.$$

Finally let ν be the unique measure on Z such that $p_J(\nu) = \nu_J$ for every $J \in \mathcal{F}$ (see [9], p. 54.)

For $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ and $1 \leq p < \infty$ we put $f = \left|\sum_{n=1}^{N} \alpha_n g_n\right|^p$. We have

$$\int_Z f(\gamma) d\nu(\gamma) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^{2N}} \left|\sum_{n=1}^{N} (x_n + iy_n)\alpha_n\right|^p e^{-(x_1^2+y_1^2)/2}$$
$$\cdots e^{-(x_N^2+y_N^2)/2} dx_1 dy_1 \cdots dx_N dy_N.$$

Suppose $(\alpha_1, \dots, \alpha_N) \neq (0, \dots, 0)$. Choose $A \in \mathcal{L}(\mathbb{C}^N)$ with $\overline{A^t} A = \mathrm{id}_{\mathbb{C}^N}$ and

$$A^{-1}\left(\frac{\left(\overline{\alpha_1}, \dots, \overline{\alpha_N}\right)}{\left(\sum_{n=1}^{N} |\alpha_n|^2\right)^{1/2}}\right) = (1, 0, \dots, 0).$$

We have

$$\int_Z f(\gamma) d\nu(\gamma) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^{2N}} \left|\left\langle (x_1 + iy_1, \dots, x_N + iy_N), \overline{A^t}\left(\overline{\alpha_1}, \dots, \overline{\alpha_N}\right)\right\rangle\right|^p$$
$$\times \exp - \left\{\frac{\left\langle (x_1 + iy_1, \dots, x_N + iy_N), (x_1 + iy_1, \dots, x_N + iy_N)\right\rangle}{2}\right\} dx_1 dy_1 \cdots dx_N dy_N$$
$$= \frac{\left(\sum_{n=1}^{N} |\alpha_n|^2\right)^{p/2}}{2\pi} \int_{\mathbb{R}^2} (x_1^2 + y_1^2)^{p/2} e^{-(x_1^2+y_1^2)/2} dx_1 dy_1 = \left(\sum_{n=1}^{N} |\alpha_n|^2\right)^{p/2} 2^{p/2}\, \Gamma\left(\frac{p+2}{2}\right).$$

The sequence (h_n) with $h_n = \dfrac{g_n}{\sqrt{2}}$ has the required properties.

Remarks. 1. This result is due to Marcinkiewicz and Zygmund ([92], p. 117) for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.
2. Our presentation is inspired by the book of Defant and Floret ([22], p. 99–100).
3. Theorem 1 implies the existence for every $1 \le p < \infty$ and every separable complex Hilbert $\mathcal{H}$ space of a linear isometry of $\mathcal{H}$ into $L^p(Z,\nu)$.

As in paragraph 3.3, X denotes a locally compact Hausdorff space, μ a positive Radon measure on X and $(V, \|\|_V)$ a complex Banach space. Let $1 \le p < \infty$. Then for $f \in \mathcal{L}^p(X,\mu)$ and $v \in V$ we have $fv \in \mathcal{L}^p_V(X,\mu)$. Let $\mathcal{N}_V$ be the set $f \in V^X$ for which $\left\{x \in G \,\middle|\, f(x) \neq 0\right\}$ is μ-negligible.

Definition 1. For $f \in L^p(X,\mu)$ and $v \in V$ we put $fv = rv + \mathcal{N}_V$ with $r \in f$.

Lemma 2. *Let X be a locally compact Hausdorff space, μ a positive Radon measure on X, $(V, \|\|_V)$ a complex Banach space and $1 \le p < \infty$.*

1. *For every $f \in L^p(X,\mu)$ and every $v \in V$ we have $fv \in L^p_V(X,\mu)$, moreover $\|fv\|_p = \|f\|_p \|v\|_V$.*
2. *Let $v_1, \ldots, v_N$ be linearly independent in V and $\left\{f_1, \ldots, f_N\right\}$ subset of $L^p(X,\mu)$. If $\sum_{n=1}^{N} f_n v_n = 0$, then $f_1 = \cdots = f_N = 0$.*

Lemma 3. *Let:*

1. *X be a locally compact Hausdorff space,*
2. *μ a positive Radon measure on X,*
3. *$1 \le p < \infty$,*
4. *$f_1, \ldots, f_N \in L^p(X,\mu)$,*
5. *$\mathcal{H}$ a complex Hilbert space,*
6. *$\left\{v_1, \ldots, v_N\right\}$ an orthonormal subset of $\mathcal{H}$,*
7. *Z the topological space of Theorem 1,*
8. *ν the measure on Z of Theorem 1,*
9. *$h_1, \ldots, h_N$ the continuous functions on Z of Theorem 1. Then*

$$\left\| \sum_{n=1}^{N} f_n v_n \right\|_p^p = \frac{1}{\Gamma\left(\dfrac{p+2}{2}\right)} \int_Z \left\| \sum_{n=1}^{N} h_n(\gamma) f_n \right\|_p^p d\nu(\gamma).$$

Proof. For $1 \le n \le N$ let $r_n \in f_n$. Then

$$N_p\left(\sum_{n=1}^{N} r_n v_n\right)^p = \int_X \left\| \sum_{n=1}^{N} r_n(x) v_n \right\|_{\mathcal{H}}^p d\mu(x).$$

For $x \in X$ we have, by Theorem 1

$$\left\| \sum_{n=1}^{N} r_n(x) v_n \right\|_{\mathcal{H}} = \frac{1}{\Gamma\left(\frac{p+2}{2}\right)^{1/p}} \left(\int_Z \left| \sum_{n=1}^{N} r_n(x) h_n(\gamma) \right|^p d\nu(\gamma) \right)^{1/p}.$$

Consequently

$$\left\| \sum_{n=1}^{N} f_n v_n \right\|_p^p = \int_X^* \frac{1}{\Gamma\left(\frac{p+2}{2}\right)} \left(\int_Z \left| \sum_{n=1}^{N} r_n(x) h_n(\gamma) \right|^p d\nu(\gamma) \right) d\mu(x).$$

The function $\sum_{n=1}^{N} r_n \otimes h_n$ being $\mu \otimes \nu$-measurable and $\mu \otimes \nu$-moderated, we have

$$\int_X^* \left(\int_Z \left| \sum_{n=1}^{N} r_n(x) h_n(\gamma) \right|^p d\nu(\gamma) \right) d\mu(x) = \int_Z \left(\int_X \left| \sum_{n=1}^{N} h_n(\gamma) r_n(x) \right|^p d\mu(x) \right) d\nu(\gamma)$$
$$= \int_Z \left\| \sum_{n=1}^{N} h_n(\gamma) f_n \right\|_p^p d\nu(\gamma).$$

Definition 2. Let X be a locally compact Hausdorff space, μ a positive Radon measure on X, $(V, \|\|_V)$ a complex Banach space and $1 \le p < \infty$. We denote by $\mathcal{E}_V^p$ the linear subspace of $L_V^p(X, \mu)$ generated by the set $\left\{ f v \,\middle|\, f \in L^p(X, \mu), v \in V \right\}$.

Lemma 4. *Let X be a locally compact Hausdorff space, μ a positive Radon measure on X, $(V, \|\|_V)$ a complex Banach space, $1 \le p < \infty$ and T a linear endomorphism of the vector space $L^p(X, \mu)$. Then there is a unique linear map U of the vector space $\mathcal{E}_V^p$ into $L_V^p(X, \mu)$ such that $U(fv) = T(f)v$ for every $f \in L^p(X, \mu)$ and every $v \in V$.*

Definition 3. The map U is denoted T_V.

Lemma 5. *Let X be a locally compact Hausdorff space, μ a positive Radon measure on X, $(\mathcal{H}, \|\|_{\mathcal{H}})$ a complex Hilbert space, $1 \le p < \infty$ and $T \in \mathcal{L}(L^p(X, \mu))$. For every $f \in \mathcal{E}_{\mathcal{H}}^p$ we then have $\|T_{\mathcal{H}} f\|_p \le \|\!|T|\!\|_p \, \|f\|_p$.*

Proof. Let $f \in \mathcal{E}_{\mathcal{H}}^p$. There is $\{v_1, \ldots, v_N\}$, an orthonormal subset of $\mathcal{H}$, and $f_1, \ldots, f_N \in L^p(X, \mu)$ such that $f = \sum_{n=1}^{N} f_n v_n$. By Lemma 3 we have

$$\|T_{\mathcal{H}} f\|_p^p = \frac{1}{\Gamma\left(\dfrac{p+2}{2}\right)} \int_Z \left\| \sum_{n=1}^N h_n(\gamma) T f_n \right\|_p^p d\nu(\gamma).$$

But for every $\gamma \in Z$

$$\left\| \sum_{n=1}^N h_n(\gamma) T f_n \right\|_p \leq \|\!|T|\!\|_p \left\| \sum_{n=1}^N h_n(\gamma) f_n \right\|_p$$

and therefore

$$\|T_{\mathcal{H}} f\|_p^p \leq \frac{\|\!|T|\!\|_p^p}{\Gamma\left(\dfrac{p+2}{2}\right)} \int_Z \left\| \sum_{j=1}^N h_n(\gamma) f_n \right\|_p^p d\nu(\gamma).$$

A second application of Lemma 3 gives the statement.

Definition 4. Let X be a locally compact Hausdorff space, μ a positive Radon measure on X, $(\mathcal{H}, \|\,\|_{\mathcal{H}})$ a complex Hilbert space, $1 \leq p < \infty$ and $T \in \mathcal{L}(L^p(X,\mu))$. The unique continuous extension of $T_{\mathcal{H}}$ from $\mathcal{E}^p_{\mathcal{H}}$ to $L^p_{\mathcal{H}}(X,\mu)$ is also denoted $T_{\mathcal{H}}$.

Theorem 6. *Let X be a locally compact Hausdorff space, μ a positive Radon measure on X, $\mathcal{H}$ a nonzero complex Hilbert space and $1 \leq p < \infty$. Then the map $T \mapsto T_{\mathcal{H}}$ is a linear isometry of the Banach space $\mathcal{L}(L^p(X,\mu))$ into the Banach space $\mathcal{L}(L^p_{\mathcal{H}}(X,\mu))$.*

Proof. Let $f \in L^p(X,\mu)$, $v \in \mathcal{H}$ with $\|f\|_p = 1$ and $\|v\|_{\mathcal{H}} = 1$. Then $\|fv\|_p = 1$, therefore $\|\!|T_{\mathcal{H}}|\!\|_p \geq \|T_{\mathcal{H}}(fv)\|_p = \|T(f)v\|_p = \|Tf\|_p \|v\|_{\mathcal{H}} = \|Tf\|_p$. This implies $\|\!|T_{\mathcal{H}}|\!\|_p \geq \|\!|T|\!\|_p$.

Remark. For $X = [a,b]$, μ the Lebesgue measure and $\mathcal{H} = \mathbb{R}^N$ with the usual scalar product, one gets the following result:

$$\left\| \left(\sum_{n=1}^N (T\varphi_n)^2 \right)^{1/2} \right\|_p \leq \|\!|T|\!\|_p \left\| \left(\sum_{n=1}^N \varphi_n^2 \right)^{1/2} \right\|_p$$

for $T \in \mathcal{L}(L^p_{\mathbb{R}}([a,b]))$ and $\varphi_1, \ldots, \varphi_N \in L^p_{\mathbb{R}}([a,b])$. This inequality is due to Marcinkiewics and Zygmund ([92], Théorème 1, p. 116). Using a result of Paley concerning the functions of Rademacher ([99], p. 250 Corollary), Marcinkiewics and Zygmund obtained at first the following weaker inequality ([92], p. 115):

$$\left\| \left(\sum_{n=1}^N (T\varphi_n)^2 \right)^{1/2} \right\|_p \leq C_p \|\!|T|\!\|_p \left\| \left(\sum_{n=1}^N \varphi_n^2 \right)^{1/2} \right\|_p$$

with a constant C_p depending only on p and such that $\lim_{p\to\infty} C_p = \infty$. For complex functions see Zygmund [120] and [121], Vol. II (2.10) Lemma, p. 224.

We generalize a definition of Sect. 1.1.

Definition 5. Let X be a locally compact Hausdorff space, μ a positive Radon measure on X, $(\mathcal{H}, <>)$ a complex Hilbert space, $1 < p < \infty$ $f \in \mathcal{L}^p_{\mathcal{H}}(X,\mu)$ and $g \in \mathcal{L}^{p'}_{\mathcal{H}}(X,\mu)$. We set:

$$\Big\langle [f], [g] \Big\rangle = \int_X \big\langle f(x), g(x) \big\rangle d\mu(x).$$

The following proposition is straightforward.

Proposition 7. *Let X be a locally compact Hausdorff space, μ a positive Radon measure on X, $(\mathcal{H}, <>)$ a complex Hilbert space, $1 < p < \infty$, $f \in L^p(X,\mu)$, $g \in L^{p'}(X,\mu)$ and $v, w \in \mathcal{H}$. Then $\big\langle fv, gw \big\rangle = \big\langle f, g \big\rangle \big\langle v, w \big\rangle$.*

8.2 An Integral Formula for $\left(T\lambda_G^p(\alpha)\right)_{L^2(G)}$

Definition 1. Let X be a set. For $F \in \mathbb{C}^{X\times X}$ and $x \in X$, we denote by $\omega(F)(x)$, the map of X into $\mathbb{C}$ defined by $y \mapsto F(x,y)$.

Let G be a locally compact group, $F \in C_{00}(G \times G)$ and $x \in G$. We have $\Big[\omega(F)(x)\Big] \in L^2(G)$ where

$$\Big[\omega(F)(x)\Big] = \Big\{ g \in \mathbb{C}^G \Big| g(y) = \omega(F)(x)(y) \quad m_G\text{-almost everywhere} \Big\}.$$

Lemma 1. *Let G be a locally compact group, $1 < p < \infty$ and $F \in C_{00}(G \times G)$. Then $x \mapsto \Big[\omega(F)(x)\Big]$ belongs to the space $\mathcal{L}^p(G; L^2(G))$. If we set*

$$f(x) = \Big[\omega(F)(x)\Big]$$

we get

$$N_p(f) = \left(\int_G \left(\int_G |F(x,y)|^2 dy \right)^{p/2} dx \right)^{1/p}.$$

Definition 2. Let G be a locally compact group, $1 < p < \infty$ and $F \in C_{00}(G\times G)$. We denote by $\varpi(F)$ the element of $L^p(G; L^2(G))$ $f + \mathcal{N}_{L^2(G)}$ where

$$f(x) = \left[\omega(F)(x)\right]$$

for $x \in G$.

Definition 3. Let G be a group and $F \in \mathbb{C}^{G\times G}$. We put $\tau(F)(x,y) = F(x, x^{-1}y)$ for every $x, y \in G$.

Lemma 2. *Let G be a locally compact group and* $1 < p < \infty$. *Then:*

1. ϖ *is a linear injective map of* $C_{00}(G \times G)$ *into* $L^p\left(G; L^2(G)\right)$,
2. *For every* $F \in C_{00}(G \times G)$ *we have*

$$\|\varpi(F)\|_p = \|\varpi(\tau(F))\|_p = \left(\int_G \left(\int_G |F(x,y)|^2 dy\right)^{p/2} dx\right)^{1/p},$$

3. *For every* $r, s \in C_{00}(G)$ *we also have* $\left\|\varpi(r \otimes s)\right\|_p = N_p(r)N_2(s)$.

Lemma 3. *Let G be a locally compact group,* $1 < p < \infty$, $T \in CV_p(G)$, $\alpha \in \mathcal{M}_{00}^{\infty}(G)$, $g \in T\left[\Delta_G^{1/p'}\alpha\right]$ *and* $F,\ F' \in C_{00}(G \times G)$. *Then:*

$$\left\langle \left(T\lambda_G^p(\overline{\alpha^*})\right)_{L^2(G)} \varpi(F), \varpi(F')\right\rangle = \int\int_{G\times G} \left(\int_G (\tau_p g)(y)\Delta_G(y)^{1/p} F(xy,z)dy\right) \overline{F'(x,z)}dxdz.$$

Proof. At first we prove the assertion for

$$F = \sum_{i=1}^{m} r_i \otimes \xi_i \quad and \quad F' = \sum_{j=1}^{n} s_j \otimes \eta_j$$

with $r_1, \ldots, r_m,\ \xi_1, \ldots, \xi_m,\ s_1, \ldots, s_n, \eta_1, \ldots, \eta_n \in C_{00}(G)$. It suffices to prove this relation for $F = r \otimes \xi$ and $F' = s \otimes \eta$ with $r, \xi, s, \eta \in C_{00}(G)$.

We have

$$\left\langle \left(T\lambda_G^p(\overline{\alpha^*})\right)_{L^2(G)} \varpi(F), \varpi(F')\right\rangle = \int_G \xi(z)\overline{\eta(z)}dz \int_G \left(\int_G (\tau_p g)(y) r(xy)\Delta_G(y)^{1/p} dy\right) \overline{s(x)}dx$$

$$= \int\int_{G\times G} \left(\int_G (\tau_p g)(y)\Delta_G(y)^{1/p} F(xy,z)dy\right) \overline{F'(x,z)}dxdz.$$

To prove the lemma for $F \in C_{00}(G \times G)$ and

$$F' = \sum_{j=1}^{m} s_j \otimes \eta_j$$

with $s_1, \ldots, s_n, \eta_1, \ldots, \eta_n \in C_{00}(G)$, it suffices to verify it for $F' = s \otimes \eta$ with $s, \eta \in C_{00}(G)$.

Let $\varepsilon > 0$. There is:

1. K, L compact subsets of G,
2. $r_1, \ldots, r_m, \xi_1, \ldots, \xi_m \in C_{00}(G)$, with $\operatorname{supp} F \subset K \times L$, $\operatorname{supp} r_j \subset K$, $\operatorname{supp} \xi_j \subset L$ for $1 \le j \le m$ and

$$\left\| F - \sum_{j=1}^{m} r_j \otimes \xi_j \right\|_\infty < \delta$$

where

$$0 < \delta < \frac{\varepsilon}{1 + A + B} \quad \text{with} \quad A = |||T|||_p \, N_1(\alpha) m(L)^{1/2} m(K)^{1/p} N_{p'}(s) N_2(\eta)$$

and

$$B = N_p(g) \left(\int_{(\operatorname{supp} s)^{-1} K} \Delta_G(y)^{p'/p} dy \right)^{1/p'} m(\operatorname{supp} s)^{1/2} m(\operatorname{supp} \eta)^{1/2} N_2(s) N_2(\eta).$$

Put

$$F'' = \sum_{j=1}^{m} r_j \otimes \xi_j$$

and

$$C = \left| \left\langle \left(T\lambda_G^p(\overline{\alpha^*}) \right)_{L^2(G)} \varpi(F), \varpi(F') \right\rangle - \int\int_{G \times G} \left(\int_G (\tau_p g)(y) \Delta_G(y)^{1/p} F(xy, z) dy \right) \overline{F'(x,z)} dx dz \right|.$$

Then

$$C \le \left| \left\langle \left(T\lambda_G^p(\overline{\alpha^*}) \right)_{L^2(G)} \varpi(F), \varpi(F') \right\rangle - \left\langle \left(T\lambda_G^p(\overline{\alpha^*}) \right)_{L^2(G)} \varpi(F''), \varpi(F') \right\rangle \right|$$
$$+ \left| \left\langle \left(T\lambda_G^p(\overline{\alpha^*}) \right)_{L^2(G)} \varpi(F''), \varpi(F') \right\rangle - \int\int_{G \times G} \left(\int_G (\tau_p g)(y) \Delta_G(y)^{1/p} F(xy, z) dy \right) \overline{F'(x,z)} dx dz \right|$$

$$
\begin{aligned}
&= \left|\left\langle \left(T\lambda_G^p(\overline{\alpha^*})\right)_{L^2(G)}\ \varpi(F-F''), \varpi(F')\right\rangle\right| \\
&\quad + \left|\iint\limits_{G\times G}\left(\int\limits_G (\tau_p g)(y)\Delta_G(y)^{1/p}(F''(xy,z)-F(xy,z))dy\right)\overline{F'(x,z)}dxdz\right| \\
&\le \left|\!\left|\!\left|(T\lambda_G^p(\overline{\alpha^*}))_{L^2(G)}\right|\!\right|\!\right|_p \|\varpi(F-F'')\|_p\,\|\varpi(F')\|_p \\
&\quad + \iint\limits_{G\times G}\left(\int\limits_G |\tau_p g)(y)|\Delta_G(y)^{1/p}\left|F''(xy,z)-F(xy,z)\right|dy\right)\left|F'(x,z)\right|dxdz.
\end{aligned}
$$

But, according to Theorem 6 of Sect. 8.1, we have

$$
\left|\!\left|\!\left|\left(T\lambda_G^p(\overline{\alpha^*})\right)_{L^2(G)}\right|\!\right|\!\right|_p = \left|\!\left|\!\left|T\lambda_G^p(\overline{\alpha^*})\right|\!\right|\!\right|_p,
$$

and by Lemma 2

$$
\|\varpi(F-F'')\|_p = \left(\int\limits_G\left(\int\limits_G |F(x,y)-F''(x,y)|^2dy\right)^{p/2}dx\right)^{1/p} \le \delta m(L)^{1/2}m(K)^{1/p}
$$

and $\|\varpi(F')\|_{p'} = N_{p'}(s)N_2(\eta)$. Therefore

$$
\begin{aligned}
&\left|\!\left|\!\left|(T\lambda_G^p(\overline{\alpha^*}))_{L^2(G)}\right|\!\right|\!\right|_p \|\varpi(F-F'')\|_p\,\|\varpi(F')\|_{p'} \\
&\le |\!|\!|T|\!|\!|_p\, N_1(\alpha)\,\delta\, m(L)^{1/2}m(K)^{1/p}N_{p'}(s)N_2(\eta) = \delta A.
\end{aligned}
$$

Together with

$$
\begin{aligned}
&\iint\limits_{G\times G}\left(\int\limits_G |\tau_p g)(y)|\Delta_G(y)^{1/p}|F''(xy,z)-F(xy,z)|dy\right)|F'(x,z)|dxdz \\
&\le \delta\, N_p(g)\left(\int\limits_{(\operatorname{supp} s)^{-1}K}\Delta_G(y)^{p'/p}dy\right)^{1/p'} m(\operatorname{supp} s)^{1/2}m(\operatorname{supp}\eta)^{1/2}N_2(s)N_2(\eta) = \delta B,
\end{aligned}
$$

we obtain $C < \varepsilon$, and thus $C = 0$.

For the general case let:

1. $F, F' \in C_{00}(G\times G)$,
2. K, L compact subsets of G with $\operatorname{supp} F \subset K\times L$,
3. $\varepsilon > 0$,
4. K_1, L_1 compact subsets of G such that $\operatorname{supp} F' \subset K_1\times L_1$,

5. $s_1, \ldots, s_n, \eta_1, \ldots, \eta_n \in C_{00}(G)$ with $\operatorname{supp} s_j \subset K_1$, $\operatorname{supp} \eta_j \subset L_1$ for $1 \leq j \leq n$ and

$$\left\| F' - \sum_{j=1}^{n} s_j \otimes \eta_j \right\|_\infty < \delta$$

where

$$0 < \delta < \frac{\varepsilon}{1 + A + B}$$

with

$$A = \|\!|T|\!\|_p \, N_1(\alpha) m(L_1)^{1/2} m(K_1)^{1/p'} \left(\int_G \left(\int_G |F(x,y)|^2 dy \right)^{p/2} dx \right)^{1/p}$$

and

$$B = \|F\|_\infty N_p(g) \left(\int_{K_1^{-1}K} \Delta_G(y)^{p'/p} dy \right)^{1/p'} m(K_1) m(L_1).$$

Put

$$F'' = \sum_{j=1}^{n} s_j \otimes \eta_j$$

and

$$C = \Bigg| \left\langle \left(T\lambda_G^p(\overline{\alpha^*}) \right)_{L^2(G)} \varpi(F), \varpi(F') \right\rangle - \int\!\!\int_{G\times G} \left(\int_G (\tau_p g)(y) \Delta_G(y)^{1/p} F(xy,z) dy \right) \overline{F'(x,z)} dx dz \Bigg|.$$

Then we have

$$C \leq \left\|\!\left| T\lambda_G^p(\overline{\alpha^*}) \right|\!\right\|_p \|\varpi(F)\|_p \, \|\varpi(F - F'')\|_{p'} + \int\!\!\int_{G\times G} \left(\int_G |\tau_p g)(y)| \Delta_G(y)^{1/p} |F(xy,z)| dy \right) |F''(x,z) - F'(x,z)| dx dz.$$

But, similarly as before, we have

$$\left\|\!\left| T\lambda_G^p(\overline{\alpha^*}) \right|\!\right\|_p \|\varpi(F)\|_p \, \|\varpi(F - F'')\|_{p'} \leq \|\!|T|\!\|_p \, N_1(\alpha) \, \delta \, m(L_1)^{1/2} m(K_1)^{1/p'} \left(\int_G \left(\int_G |F(x,y)|^2 dy \right)^{p/2} dx \right)^{1/p},$$

and

$$\iint\limits_{G\times G}\left(\int\limits_G |\tau_p g)(y)|\Delta_G(y)^{1/p}|F(xy,z)|dy\right)|F''(x,z)-F'(x,z)|dxdz$$
$$\leq \delta\|F\|_\infty N_p(g)\left(\int\limits_{K_1^{-1}K}\Delta_G(y)^{p'/p}dy\right)^{1/p'} m(K_1)m(L_1),$$

this also implies $C<\varepsilon$, which again implies $C=0$.

Theorem 4. *Let G be a locally compact group,* $1<p<\infty$, $T\in CV_p(G)$, $\alpha\in\mathcal{M}_{00}^\infty(G)$, $g\in T\left[\Delta_G^{1/p'}\alpha\right]$ *and* $k,l,\varphi,\psi\in C_{00}(G)$. *Then*

$$\left\langle\left(T\lambda_G^p(\overline{\alpha^*})\right)_{L^2(G)}\ \varpi(\tau(F)),\varpi(\tau(F'))\right\rangle=\left\langle(\overline{k}*\check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)[\tau_p\varphi],[\tau_{p'}\psi]\right\rangle$$

with $F=(\tau_p k)\otimes\varphi$ *and* $F'=(\tau_{p'}l)\otimes\psi$.

Proof. By Lemma 1 of Sect. 5.2

$$\left\langle(\overline{k}*\check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)[\tau_p\varphi],[\tau_{p'}\psi]\right\rangle=\int\limits_G g(y)\Delta_G(y)^{-1/p'}\overline{(\overline{k}*\check{l})(y)}\overline{\left\langle\lambda_G^{p'}(\delta_y)[\tau_{p'}\psi],[\tau_p\varphi]\right\rangle}dy.$$

From

$$\left\langle\lambda_G^{p'}(\delta_y)[\tau_{p'}\psi],[\tau_p\varphi]\right\rangle=\left\langle\lambda_G^2(\delta_y)[\tau_2\psi],[\tau_2\varphi]\right\rangle=(\overline{\varphi}*\check{\psi})(y)$$

we get

$$\left\langle(\overline{k}*\check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)[\tau_p\varphi],[\tau_{p'}\psi]\right\rangle=\int\limits_G(\tau_p g)(y)\left\langle\lambda_G^p(\delta_y)[\tau_p k],[\tau_{p'}l]\right\rangle\left\langle\lambda_G^2(\delta_y)[\tau_2\varphi],[\tau_2\psi]\right\rangle dy.$$

But by Lemma 3

$$\left\langle\left(T\lambda_G^p(\overline{\alpha^*})\right)_{L^2(G)}\ \varpi(\tau(F)),\varpi(\tau(F'))\right\rangle$$
$$=\int\limits_G(\tau_p g)(y)\Delta_G(y)^{1/p}\left(\int\limits_G\left(\int\limits_G F(xy,y^{-1}x^{-1}z)\overline{F'(x,x^{-1}z)}dz\right)dx\right)dy$$
$$=\int\limits_G(\tau_p g)(y)\left\langle\lambda_G^p(\delta_y)[\tau_p k],[\tau_{p'}l]\right\rangle\left\langle\lambda_G^2(\delta_y)[\tau_2\varphi],[\tau_2\psi]\right\rangle dy.$$

8.3 $CV_2(G)$ and $CV_p(G)$ in Amenable Groups

Theorem 1. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$, $T \in CV_p(G)$ and $\varphi \in L^p(G) \cap L^2(G)$. Then:*

1. $\tau_p(uT(\tau_p\varphi)) \in L^2(G)$,
2. $\|\tau_p(uT(\tau_p\varphi))\|_2 \le \|u\|_{A_p} \, |||T|||_p \, \|\varphi\|_2$.

Proof. 1. For $T \in CV_p(G)$, $\alpha \in M_{00}^{\infty}(G)$, $g \in T\left[\Delta_G^{1/p'}\alpha\right]$ and $k, l, \varphi, \psi \in C_{00}(G)$ we have

$$\left|\left\langle(\overline{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| \le |||T\lambda_G^p(\overline{\alpha^*})|||_p \, N_p(k)N_{p'}(l)N_2(\varphi)N_2(\psi).$$

By Theorem 4 of Sect. 8.2

$$\left\langle(\overline{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle = \left\langle\left(T\lambda_G^p(\overline{\alpha^*})\right)_{L^2(G)} \varpi(\tau(F)), \varpi(\tau(F'))\right\rangle$$

with $F = (\tau_p k) \otimes \varphi$ and $F' = (\tau_{p'} l) \otimes \psi$, and by Lemma 2 of Sect. 8.2

$$\left|\left\langle(\overline{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| \le |||T\lambda_G^p(\overline{\alpha^*})|||_p \, N_p(k)N_{p'}(l)N_2(\varphi)N_2(\psi).$$

2. Next we prove that for $T \in CV_p(G)$ and $k, l, \varphi, \psi \in C_{00}(G)$

$$\left|\left\langle(\overline{k} * \check{l})T)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| \le |||T|||_p \, N_p(k)N_{p'}(l)N_2(\varphi)N_2(\psi).$$

Let $\varepsilon > 0$. By Lemma 1 of Sect. 5.3 there is $\alpha \in C_{00}(G)$ with $\alpha \ge 0$, $\int_G \alpha(y)dy = 1$ and

$$\left|\left\langle(\overline{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle - \left\langle(\overline{k} * \check{l})T[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| < \varepsilon$$

and therefore

$$\left|\left\langle(\overline{k} * \check{l})T[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| < \varepsilon + |||T|||_p \, N_p(k)N_{p'}(l)N_2(\varphi)N_2(\psi).$$

3. To finish the proof, it suffices to verify that for $T \in CV_p(G)$, $u \in A_p(G)$ and $\varphi, \psi \in C_{00}(G)$ we have

$$\left|\left\langle (uT)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| \leq |\!|\!| T |\!|\!|_p \, \|u\|_{A_p} N_2(\varphi) N_2(\psi).$$

Let $\varepsilon > 0$. There exist sequences (k_n) and (l_n) of $C_{00}(G)$ with

$$u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n$$

and

$$\sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) < \|u\|_{A_p} + \frac{\varepsilon}{1 + |\!|\!| T |\!|\!|_p N_2(\varphi) N_2(\psi)}.$$

We obtain

$$\left|\left\langle (uT)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| \leq \sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n)\, |\!|\!| T |\!|\!|_p \, N_2(\varphi) N_2(\psi) < \varepsilon + |\!|\!| T |\!|\!|_p \, \|u\|_{A_p} N_2(\varphi) N_2(\psi).$$

We are now able to extend Corollary 5 of Sect. 1.5 to the class of amenable groups.

Theorem 2. *Let G be a locally compact amenable group,* $1 < p < \infty$*,* $T \in CV_p(G)$ *and* $\varphi \in L^p(G) \cap L^2(G)$*. Then:*

1. $\tau_p T \tau_p \varphi \in L^2(G)$,
2. $\|\tau_p T \tau_p \varphi\|_2 \leq |\!|\!| T |\!|\!|_p \, \|\varphi\|_2$.

Proof. Let $\psi \in C_{00}(G)$ with $N_2(\psi) \leq 1$. Let $\varepsilon > 0$. By Theorem 2 of Sect. 5.4 there is $k, l \in C_{00}(G)$ with $N_p(k) = N_{p'}(l) = 1$ and

$$\left|\left\langle \left(\overline{k} * \check{l})T\right)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle - \left\langle T[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| < \varepsilon.$$

With the Theorem 1 we obtain

$$\left|\left\langle \tau_p T \tau_p \varphi, [\psi]\right\rangle\right| < \varepsilon + \left\|\tau_p\left((\overline{k} * \check{l})T\right)\tau_p\varphi\right\|_2 \leq \varepsilon + \|\overline{k} * \check{l}\|_{A_p} |\!|\!| T |\!|\!|_p \, \|\varphi\|_2 \leq \varepsilon + |\!|\!| T |\!|\!|_p \, \|\varphi\|_2.$$

Definition 1. Let G be a locally compact group and $1 < p < \infty$. We denote by E_p the set of all $T \in CV_p(G)$ such that:

i. $\tau_p T \tau_p \varphi \in L^2(G)$ for every $\varphi \in L^p(G) \cap L^2(G)$,
ii. There is $C > 0$ such that $\|\tau_p T \tau_p \varphi\|_2 \leq C\|\varphi\|_2$ for every $\varphi \in L^p(G) \cap L^2(G)$.

Corollary 3. *Let G be a locally compact group and* $1 < p < \infty$*. Then:*

1. *E_p is a subalgebra of $CV_p(G)$,*
2. *$\lambda_G^p(M^1(G)) \subset E_p$,*
3. *$A_p(G)CV_p(G) \subset E_p$.*

Lemma 4. *Let G be a locally compact group, $1 < p < \infty$ and $T \in E_p$. Then there is an unique operator $S \in \mathcal{L}(L^2(G))$ with $S\varphi = \tau_p T \tau_p \varphi$ for every $\varphi \in L^p(G) \cap L^2(G)$. We have $\tau_2 S \tau_2 \in CV_2(G)$.*

Definition 2. Let G be a locally compact group, $1 < p < \infty$, $T \in E_p$ and S as in Lemma 4. We put $\alpha_p(T) = \tau_2 S \tau_2$.

Theorem 5. *Let G be a locally compact group and $1 < p < \infty$. Then:*

1. *α_p is an injective homomorphism of the algebra E_p into $CV_2(G)$,*
2. *$\alpha_p(\lambda_G^p(\mu)) = \lambda_G^2(\mu)$ for $\mu \in M^1(G)$.*

In the following two theorems we extend Theorem 4 and Corollary 6 of Sect. 1.5 to the whole class of locally compact amenable groups.

Theorem 6. *Let G be an amenable locally compact group and $1 < p < \infty$. Then:*

1. *α_p is an injective homomorphism of the Banach algebra $CV_p(G)$ into $CV_2(G)$,*
2. *$\left|\!\left|\!\left|\alpha_p(T)\right|\!\right|\!\right|_2 \le |\!|\!| T |\!|\!|_p$ for every $T \in CV_p(G)$.*

Theorem 7. *Let G be an amenable locally compact group and $1 < p < \infty$. For every bounded Radon measure μ on G we have $\left|\!\left|\!\left|\lambda_G^2(\mu)\right|\!\right|\!\right|_2 \le \left|\!\left|\!\left|\lambda_G^p(\mu)\right|\!\right|\!\right|_p$.*

Theorem 8. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_2(G)$ and $v \in A_p(G)$. Then:*

1. *$uv \in A_p(G)$ and $\|uv\|_{A_p} \le \|u\|_{A_2}\|v\|_{A_p}$, i.e. $A_p(G)$ is a Banach module on $A_2(G)$*
2. *$\left\langle uv, T\right\rangle_{A_p, PM_p} = \left\langle u, \alpha_p(vT)\right\rangle_{A_2, PM_2}$ for every $T \in PM_p(G)$.*

Proof. Let $\Big((k_n), (l_n)\Big) \in \mathcal{A}_2(G)$ with

$$u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n.$$

For every $F \in A_p(G)'$ we put

$$\omega(F) = \sum_{n=1}^{\infty} \overline{\left\langle \alpha_p\Big(v\left(\Psi_G^p\right)^{-1}(F)\Big)[\tau_2 k_n], [\tau_2 l_n]\right\rangle}$$

where Ψ_G^p is defined in Sect. 4.1 (Definition 3).

Let F be an element of $A_p(G)'$ and $(F_i)_{i \in I}$ a net of $A_p(G)'$ such that $\lim F_i = F$ for the topology $\sigma(A'_p, A_p)$ and with $\|F_i\|_{A'_p} \le C$ for every $i \in I$ for some $C > 0$. We claim that $\lim \omega(F_i) = \omega(F)$. By Theorem 6 of Sect. 4.1 we have

$$\lim \left(\Psi_G^p\right)^{-1}(F_i) = \left(\Psi_G^p\right)^{-1}(F)$$

for the ultraweak topology and moreover $\left|\left|\left|\left(\Psi_G^p\right)^{-1}(F_i)\right|\right|\right|_p \leq C$ for every $i \in I$. This implies that

$$\lim v\Big(\left(\Psi_G^p\right)^{-1}(F_i)\Big) = v\Big(\left(\Psi_G^p\right)^{-1}(F)\Big)$$

for the ultraweak topology and that

$$\left|\left|\left|v\left(\left(\Psi_G^p\right)^{-1}(F_i)\right)\right|\right|\right|_p \leq \|v\|_{A_p} \left|\left|\left|\left(\Psi_G^p\right)^{-1}(F_i)\right|\right|\right|_p \leq C\|v\|_{A_p}.$$

For $r, s \in M_{00}^\infty(G)$ we have

$$\lim\left\langle v\left(\left(\Psi_G^p\right)^{-1}(F_i)\right)[\tau_p r], [\tau_{p'} s]\right\rangle = \left\langle v\Big(\left(\Psi_G^p\right)^{-1}(F)\Big)[\tau_p r], [\tau_{p'} s]\right\rangle.$$

From $[r] \in L^p(G) \cap L^2(G)$, $v\Big(\left(\Psi_G^p\right)^{-1}(F_i)\Big), v\Big(\left(\Psi_G^p\right)^{-1}(F)\Big) \in E_p$ we obtain

$$\lim\left\langle \alpha_p\left(v\Big(\left(\Psi_G^p\right)^{-1}(F_i)\Big)\right)[\tau_2 r], [\tau_2 s]\right\rangle = \left\langle \alpha_p\left(v\Big(\left(\Psi_G^p\right)^{-1}(F)\Big)\right)[\tau_2 r], [\tau_2 s]\right\rangle$$

with for every $i \in I$

$$\left|\left|\left|\alpha_p\left(v\Big(\left(\Psi_G^p\right)^{-1}(F_i)\Big)\right)\right|\right|\right|_2 \leq \left|\left|\left|v\Big(\left(\Psi_G^p\right)^{-1}(F_i)\Big)\right|\right|\right|_p \leq C\|v\|_{A_p}.$$

Consequently

$$\lim \alpha_p\left(v\Big(\left(\Psi_G^p\right)^{-1}(F_i)\Big)\right) = \alpha_p\left(v\Big(\left(\Psi_G^p\right)^{-1}(F)\Big)\right)$$

for the ultraweak topology and consequently $\lim \omega(F_i) = \omega(F)$. According to Theorem 6 of Sect. 4.2 there is $w \in A_p(G)$ with $\omega(F) = F(w)$ for every $F \in A_p(G)'$, for every $T \in PM_p(G)$ we thus have $\langle w, T\rangle_{A_p,PM_p} = \langle u, \alpha_p(vT)\rangle_{A_2,PM_2}$. In particular (taking into account Proposition 5 of Sect. 5.3) for every $\mu \in M^1(G)$

$$\langle w, \lambda_G^p(\mu)\rangle_{A_p,PM_p} = \left\langle u, \alpha_p\left(v\lambda_G^p(\mu)\right)\right\rangle_{A_2,PM_2} = \left\langle u, \lambda_G^2(\tilde{v}\mu)\right\rangle_{A_2,PM_2}$$

and therefore $w = uv$. This implies indeed $uv \in A_p(G)$. Moreover we have for every $T \in PM_p(G)$

$$\left|\langle uv, T\rangle_{A_p,PM_p}\right| = \left|\langle u, \alpha_p(vT)\rangle_{A_2,PM_2}\right| \leq \|u\|_{A_2} |||\alpha(vT)|||_2 \leq \|u\|_{A_2}\|v\|_{A_p} |||T|||_p,$$

whence $\|uv\|_{A_p} \leq \|u\|_{A_2}\|v\|_{A_p}$.

Theorem 9. *Let G be an amenable locally compact group and $1 < p < \infty$. Then:*

1. $A_2(G) \subset A_p(G)$,
2. *For every $u \in A_2(G)$ one has $\|u\|_{A_p} \leq \|u\|_{A_2}$.*

Proof. Let $u \in A_2(G)$ and $\Big((k_n), (l_n)\Big) \in \mathcal{A}_2(G)$ with

$$u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n.$$

For every $F \in A_p(G)'$ we put

$$\omega(F) = \sum_{n=1}^{\infty} \overline{\left\langle \alpha_p\Big(\big(\Psi_G^p\big)^{-1}(F)\Big)[\tau_2 k_n], [\tau_2 l_n] \right\rangle}.$$

There is $v \in A_p(G)$ with $v \in A_p(G)$ with $\omega(F) = F(v)$ for every $F \in A_p(G)'$, as in the proof of Theorem 8 $u = v$ and $\|u\|_{A_p} \leq \|u\|_{A_2}$.

Remarks. 1. Theorem 9 generalizes Theorem 7 of Sect. 4.2.
2. Theorem 7 is not trivial even for a finite group!
3. For G amenable, the map α_p is the adjoint of the inclusion of $A_2(G)$ into $A_p(G)$.
4. Theorems 6, 7 and 9 are due to Herz ([59], Theorems B and C, p. 72). Nota bene, the second sentence of Theorem C should be read: "Dually there is a morphism $CV_p(G) \to CV_q(G)$, i.e. convolution operators on $L_p(G;\mathbb{C})$ are convolution operators on $L_q(G;\mathbb{C})$ with contraction of norms". The complete proof of Theorem C is given by Herz and Rivière in [65] (Corollary p. 512).
5. For G compact and $p > 2$, Theorems 6 and 7 were obtained earlier (1966) by Figà-Talamanca and Rider ([47], p. 511 line 2 from below) with the inequalities $\left|\!\left|\!\left|\alpha_p(T)\right|\!\right|\!\right|_2 \leq C_p \left|\!\left|\!\left|T\right|\!\right|\!\right|_p$ for $T \in CV_p(G)$ and $\|u\|_{A_p} \leq C_p\|u\|_{A_2}$ for $u \in A_2(G)$, with an explicit constant C_p depending only of p (and not of the group G). Figà-Talamanca and Rider's contribution is not mentioned in [59] and [65].
6. The author is grateful to professor Gerhard Racher for the communication of unpublished notes and explanations in relation with Sects. 8.1 and 8.2.

Notes

In these notes we give additional comments and supplementary bibliographical references.

Chapter 1

For the older literature and in particular for $CV_p(\mathbb{T})$ we refer to Chap. 16 of Edwards book on Fourier series [39]. For $\mathbb{R}^n$ an important paper on the subject is due to Hörmander [68]. It contains in particular various generalizations to $\mathbb{R}^n$ of Marcel Riesz's Theorem.

For $p = 1$ and G, an arbitrary locally compact group, the space of operators which corresponds to $CV_p(G)$ is precisely $M^1(G)$. This a famous result due to Wendel [117].

For $p = \infty$ the question is more subtle. Our Theorem 5 of Sect. 1.2, is no more true. Indeed, according to Stafney [113] there is on $L^\infty(\mathbb{T})$ an non-zero linear continuous invariant functional M which is zero on all continuous functions, consequently the operator $\varphi \mapsto M(\varphi)\varphi$ does not commute with the action of L^1. The author is indebted to the referee for communication of this fact which was unknown to him. For other informations on the case $p = \infty$ see [73], p. 74–78.

To justify the Remark 2 of Theorem 8 in Sect. 1.2 we used the following fact: for amenable groups the convolution norm in L^p of a positive measure is exactly its total mass. One should mention that this result fails for non amenable groups: if for some $1 < p < \infty$ the convolution norm in L^p of every positive $f \in L^1(G)$ coincides with $\int f(x)dx$ then G is amenable ([105], Theorem 8.3.19, p. 241). For specific non amenable groups more precise results have been obtained. In [85] Lohoué constructed, for any $1 < p_0 < \infty$, a positive measure on $SL_2(\mathbb{R})$ which convolves L^{p_0} but does not convolve any other L^p! Similar results were also obtained latter by Nebbia for the group of isometries of a homogenous tree [97]. According to Leinert

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[75] every l^2 function on $\mathbb{F}_2$ supported by a (possibly infinite) free set defines a bounded convolution operator on l^2.

The fact that, for $G = SL_2(\mathbb{R})$ (see Remark 4 of Theorem 8 in Sect. 1.2) every function in $L^p(G)$ convolves $L^2(G)$ for $1 < p < 2$, requires, besides the Plancherel formula of Harish-Chandra, the study of a new class of infinite dimensional representations obtained by "analytic continuation" of the principal series and "some vigorous" classical Fourier analysis. These representations acts on a fixed Hilbert space, they are indexed by a complex parameter and depend analytically on the parameter, included among them are the complementary series [72]. A locally compact unimodular group G is said to have the "Kunze-Stein property" if $L^2(G) * L^p(G) \subset L^2(G)$ for every $1 < p < 2$ [77]. Clearly compact groups have this property but noncompact amenable groups don't. In 1970 Lipsman [78] proved the following result: let G be a noncompact locally compact connected unimodular group. Then G has the Kunze-Stein property if and only if G has a compact normal subgroup H such that (1) G/H is a connected semisimple Lie group with finite center and (2) G/H has the Kunze-Stein property. Finally Cowling proved that connected semisimple Lie groups with finite center have the Kunze-Stein property [17].

Oberlin [98], with the aid of a computer, proved that for a certain function f, defined on the dihedral group D_4 of height elements,

$$\left\|\left|\lambda_{D_4}^4(f)\right\|\right|_4 \neq \left\|\left|\lambda_{D_4}^{4/3}(f)\right\|\right|_{4/3}.$$

As mentioned in Sect. 1.4 (Remark after the Corollary 7), Herz proved a similar statement for every finite nonabelian group [63]. By the Corollary 3 of Chap. 7, Sect. 7.3 it follows that, for a locally compact group G having a finite non-abelian subgroup, and for every $p \neq 2$ there is a bounded measure μ with $\left\|\left|\lambda_G^p(\mu)\right\|\right|_p \neq \left\|\left|\lambda_G^{p'}(\mu)\right\|\right|_{p'}$. A consequence of [85], is that for $SL_2(\mathbb{R})$ for every $p \neq 2$ there is a positive measure which convolves L^p but not $L^{p'}$. In [63] and [64] Herz proved the following theorem: for each non-abelian nilpotent Lie group G and each p with $2 < p < \infty$ there exists a sequence (k_n) of $L^1(G)$ such that $\left\|\left|\lambda_G^p(k_n)\right\|\right|_p \leq 1$ while $\left\|\left|\lambda_G^{p'}(k_n)\right\|\right|_{p'} \to \infty$. See Dooley et al. [37] for refinements (one could require for instance that the k_n have their support in a given neighborhood of e) and various improvements on these questions.

Chapter 3

The sesquilinear map $(f, g) \mapsto \overline{f} * \check{g}$ of $L^p(G) \times L^{p'}(G)$ into $C_0(G)$ factorizes through a linear contraction P of $L^p(G)\widehat{\otimes}L^{p'}(G)$ into $C_0(G)$ where $L^p(G)\widehat{\otimes}L^{p'}(G)$ is the projective tensorproduct of $L^p(G)$ with $L^{p'}(G)$. Then clearly $A_p(G)$ coincides with the quotient Banach space $L^p(G)\widehat{\otimes}L^{p'}(G)/KerP$. This was used by Herz for his proof of Corollary 5 in Sect. 3.3.

It is possible to repeat the Sects. 3.1 and 4.1 replacing the regular representation in L^2 by an arbitrary continuous unitary representation π of G. One gets a Banach space, denoted A_π, of nice uniformly continuous bounded functions on the group G. See Arsac [4]. He proved, among other things, that A_π is a Banach algebra if and only if the representation $\pi \otimes \pi$ is quasi equivalent to π. In relation with the Kunze-Stein property, Herz considered such an A_π, choosing for π the representation induced by the identity representation from a certain closed subgroup H of a connected semi-simple Lie group of finite centre [58]. The group H is AN, where $G = KAN$ is the Iwasawa decomposition of G. He proved ("principe de majoration") that for $u \in A_2(G)$ and $\varepsilon > 0$ there is $v \in A_\pi$ with $\|v\|_{A_\pi} < \|u\|_{A_2} + \varepsilon$ and $u(x) \le v(x)$ for every $x \in G$. This was used in [17].

Let $B(G)$ be the vector space of the coefficients of all continuous unitary representations of G. Generalizing the fact that $A_2(G)$ is an algebra, P. Eymard ([41]) proved that each u of $B(G)$ "multiplies" $A_2(G)$ i.e. $uA_2(G) \subset A_2(G)$. If G is abelian the converse is true: if u multiplies $A_2(G)$ then $u \in B(G)$ (see [107], p. 73). This holds also for amenable G [27]. Losert proved that if G is non amenable then there are multipliers of $A_2(G)$ which are not in $B(G)$ [86].

For a better understanding of the multipliers of $A_2(G)$ (and also of $A_p(G)$) we need to consider an important class of convolution operators. We denote by $PF_p(G)$ the norm closure in $CV_p(G)$ of $\left\{\lambda_G^p(f) \,\middle|\, f \in \mathcal{L}^1(G)\right\}$. If G is abelian $PF_2(G)$ is isomorphic to $C_0(\widehat{G})$. See [70], p. 45 for $\mathbb{T}$. The Banach algebra $PF_2(\mathbb{T})$ is isomorphic to $c_0(\mathbb{Z})$ (and $CV_2(\mathbb{T})$ to $l^\infty(\mathbb{Z})$). Kahane call the elements of $CV_2(\mathbb{T})$ "pseudomeasures" and the elements of $PF_2(\mathbb{T})$ "pseudofunctions". For $p \neq 2$ the algebra $PF_p(\mathbb{R}^n)$ has been considered by Hörmander in [68], p. 111. Answering a question raised by Hörmander, Figà-Talamanca and Gaudry proved that for every $p \neq 2$ there is a $T \in CV_p(\mathbb{R}^n)$ with $\widehat{T} \in C_0(\mathbb{R}^n)$ such that $T \notin PF_p(\mathbb{R}^n)$ ([46]). They obtained a similar statement for $\mathbb{T}^n$.

For a general locally compact group $PF_2(G)$ is called the reduced C^*- algebra of G. According to Eymard [41] $PF_2(G)$ coincides with the full C^*- algebra of G if and only if G is amenable. Recall (see again [41]) that the dual of the full C^*-algebra of G is precisely $B(G)$. Observe that if G has the Kunze-Stein property then clearly $L^p(G) \subset PF_2(G)$ for every $1 < p < 2$ and $A_2(G) \subset L^q(G)$ for every $q > 2$. If G is abelian then, according to Lohoué, the space of all multipliers of $A_p(G)$ is the dual of the Banach space $PF_p(G)$ for $p > 1$ ([79], Théorème 1). In [62] Herz proved the following: (1) for an arbitrary locally compact group G and for $p > 1$ every function in the dual of $PF_p(G)$ multiplies $A_p(G)$; (2) if G is amenable the dual of $PF_p(G)$ coincides with the space of all multipliers of $A_p(G)$. To obtain these results, C. Herz introduces two interesting commutative unital Banach algebras denoted respectively $V_p(G \times G)$ and $B_p(G)$. We have $V_p(G \times G) \subset C^b(G \times G)$ and $B_p(G) \subset C^b(G)$ and in both algebras the product is simply the pointwise product. For every $k \in C_{00}(G \times G)$ we denote by T_k the operator of $L^p(G)$ having the kernel k. Then by definition $V_p(G \times G) =$

$\{u | u \in C^b(G \times G)$, there is $C > 0$ such that $|||T_{uk}|||_p \leq C\, |||T_k|||_p$ for every $k \in C_{00}(G) \otimes C_{00}(G)\}$. For $u \in V_p(G \times G)$ the norm of u is the infimum of all C. The definition of $B_p(G)$ is: $B_p(G) = \{u | u \in C(G)$ and the function on $G \times G$ $(x, y) \to u(yx^{-1})$ belongs to $V_p(G \times G)\}$, the norm of u being the norm in $V_p(G \times G)$ of the function $(x, y) \to u(yx^{-1})$. Then Herz proved the following statements: (1) $B_p(G)A_p(G) \subset A_p(G)$; (2) the dual of $PF_p(G)$ is contained in $B_p(G)$; (3) if G is amenable then $B_p(G)$ coincides with the set all multipliers of $A_p(G)$ (and thus with the dual of $PF_p(G)$). The letter V of $V_p(G \times G)$ refers to Varopoulos (see [56]). The algebra $B_p(G)$ has been used by Herz to study the asymmetry of the norms of convolution operators [64]. For G abelian Lohoué was able to relate $B_p(G)$ with $B_p(G_d)$: he proved in particular that $B_p(G) = B_p(G_d) \cap C(G)$ and more subtle results (see Théorème 3, p. 28 in [81] and Théorème 0.2.2, p. 78 in [80]). For a generalization of these results of Lohoué to all locally compact groups, see [62] and [33] (Theorem 7, p. 936). For more on $B_p(G)$ see [11, 12, 51].

Eymard proved in [41] that the dual of $PF_2(G)$ is the vector space of the coefficients of the unitary representations weakly contained in the regular representation of G. Cowling and Fendler ([19], Theorem 2) obtained a similar description of the dual of $PF_p(G)$ for every $p > 1$. They replace the unitary representations by isometric representations in certain reflexive Banach spaces of a very precise type and use the theory of ultraproducts of Banach spaces.

Chapter 4

Like L^1, $A_2(G)$ is weakly sequentially complete. This follows from the fact that $A_2(G)$ is the predual of the von Neumann algebra $CV_2(G)$ ([114], Chap. III, Corollary 5.2, p. 148). If $p \neq 2$ the question is not completely answered. If G is compact metrizable, then $A_p(G)$ is weakly sequentially complete (Lust-Piquard [88]). Lust-Piquard's proof uses the characterization (see the notes to Chap. 3) of $A_p(G)$ as a quotient of $L^p(G)\widehat{\otimes}L^{p'}(G)$. For $\mathbb{R}$ the problem seems to be open.

Hörmander proved that holomorphic functions operate on $\{\widehat{T} | T \in PF_p(\mathbb{R}^n)\}$ ([68], Theorem 1.18, p. 112). In [118] Zafran, for $G = \mathbb{R}^n$ or $G = \mathbb{T}^n$, shows the existence of $T \in CV_p(G)$ with $\widehat{T} \in C_0(\widehat{G})$ and such that the spectrum of T properly contains $\widehat{T}(\widehat{G}) \cup \{0\}$. The existence of a bounded measure having this property is the classical theorem of Wiener-Pitt ([107], Theorem 5.3.4.) See also [119].

Chapter 5

For a large class of nonamenable groups G, including $SO(1,n)$, $SU(1,n)$ and $Sp(1,n)$, the algebra $A_2(G)$ admits approximate units bounded in the norm $B_2(G)$ [20], [21].

Cowling proved in [16] that $CV_p(G) = PM_p(G)$ for every $p > 1$ for $G = SL_2(\mathbb{R})$. He obtained also this result for the above mentioned groups, proving following general result: $PM_p(G) = CV_p(G)$ if $A_2(G)$ has an approximate unit bounded in the norm of $B_2(G)$ ([18], p. 413).

Chapter 7

Let H be a closed subgroup of a locally compact abelian group G and $T \in CV_p(G)$ with $\widehat{T} \in C(\widehat{G})$, then there is $S \in CV_p(G/H)$ with $Res_{H^\perp}\widehat{T} = \widehat{S}\circ\widehat{\omega}$. One has $|||S|||_p \leq |||T|||_p$ [74], [108]. Figà-Talamanca and Gaudry, supposing $H^\perp$ discrete, obtained the following extension theorem: for every $S \in CV_p(G/H)$ there is $T \in CV_p(G)$ with $\widehat{T} \in C(\widehat{G})$, $|||T|||_p \leq K\,|||S|||_p$ and $Res_{H^\perp}\widehat{T} = S\circ\widehat{\omega}$ [45]. For $G = \mathbb{R}^n$ and $\mathbb{Z}^n = \mathbb{R}^n$ see Jodeit [69]. Cowling [15] has been able to suppress the extra condition on the discreteness of $H^\perp$: for every $S \in CV_p(G/H)$ with $\widehat{S} \in C(\widehat{G/H})$ there is $T \in CV_p(G)$ with $\widehat{T} \in C(\widehat{G})$, $|||T|||_p \leq |||S|||_p$ and $Res_{H^\perp}\widehat{T} = S\circ\widehat{\omega}$.

Instead of $\{\widehat{T} | T \in CV_p(G), \widehat{T} \in C(\widehat{G})\}$ one could consider the norm closure of the set of all convolution operators having compact support. For this class of convolution operators the preceding restriction and extension results can be generalized to noncommutative groups [2, 29, 60].

Let G be an arbitrary locally compact group, H a closed normal subgroup and F a closed subset of G/H. In analogy with a famous result of Reiter ([105], Theorem 7.3.1, p. 203), Lohoué proved, that F is locally of p-synthesis in G/H if and only if $\omega^{-1}(F)$ is locally of p-synthesis in G [82]. A closed subset F of G is said to be locally p-Ditkin in G if for every $u \in A_p(G) \cap C_{00}(G)$ vanishing on F and for every $\varepsilon > 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$ and $\|u - uv\|_{A_p} < \varepsilon$. The closed set F is said to be p-Ditkin in G if for every $u \in A_p(G)$ vanishing on F and for every $\varepsilon > 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$ and $\|u - uv\|_{A_p} < \varepsilon$. If every element u of $A_p(G)$ belongs to the closure of $uA_p(G)$ these notions coincide. The preceding theorem of Lohoué is valid also for Ditkin sets. Indeed we proved in [30] that every closed normal subgroup is locally p-Ditkin in G. This result was later extended to neutral subgroups [24] and to amenable subgroups [32]. Finally Ludwig and Turowska proved that every closed subgroup of a second countable locally compact group is locally 2-Ditkin [87].

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List of Definitions

We list here only the definitions which are systematically used throughout the book. The numbers in parentheses refer to the paragraphs where the definitions are given

Amenable groups (1.1.4)

Bruhat function (7.1)

Convolution operator (1.2)

Convolution operator associated to a bounded measure (1.2)

Duality between $A_p(G)$ and $PM_p(G)$ (4.1)

Figà-Talamanca Herz algebra (3.1), (3.3)

$CV_p(G)$ as a module on $A_p(G)$ (5.2)

Pseudomeasure (4.1)

Spectrum (6.1)

Support of a convolution operator (6.1)

Ultraweak topology (4.1)

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